Published by Institute of Physics Publishing for SISSA
Received: *May 2, 2007*
Accepted: *June 18, 2007*
Published: *July 2, 2007*

One-loop MHV rules and pure Yang-Mills

Andreas Brandhuber, Bill Spence, Gabriele Travaglini and Konstantinos Zoubos

Centre for Research in String Theory, Department of Physics,
Queen Mary, University of London,
Mile End Road, London, E1 4NS, United Kingdom
E-mail: a.brandhuber@qmul.ac.uk, w.j.spence@qmul.ac.uk, g.travaglini@qmul.ac.uk, k.zoubos@qmul.ac.uk

Abstract: It has been known for some time that the standard MHV diagram formulation of perturbative Yang-Mills theory is incomplete, as it misses rational terms in one-loop scattering amplitudes of pure Yang-Mills. We propose that certain Lorentz violating counterterms, when expressed in the field variables which give rise to standard MHV vertices, produce precisely these missing terms. These counterterms appear when Yang-Mills is treated with a regulator, introduced by Thorn and collaborators, which arises in world-sheet formulations of Yang-Mills theory in the lightcone gauge. As an illustration of our proposal, we show that a simple one-loop, two-point counterterm is the generating function for the infinite sequence of one-loop, all-plus helicity amplitudes in pure Yang-Mills, in complete agreement with known expressions.

Keywords: Duality in Gauge Field Theories, Gauge Symmetry, String Duality, Integrable Field Theories.

JHEP07(2007)002

Contents

1. Introduction

One of the success stories arising from twistor string theory [1] (see [2] for a review) has been the development of new techniques in perturbative quantum field theory. These include recursion relations [3, 4], generalised unitarity [5] and MHV methods (see [6] for a review). One of the key motivations of this work is to provide new approaches to study and derive phenomenologically relevant scattering amplitudes. In particular, this requires that one be able to deal with non-supersymmetric theories, and to include fermions, scalars, and particles with masses. A vital first step is to apply these new methods to pure Yang-Mills (YM) theory, and indeed, some of the first new results inspired by twistor string theory involved pure YM amplitudes at tree- [7 − 14] and one-loop [15] level.

A recalcitrant issue in this work is the derivation of rational terms in quantum amplitudes. Unitarity-based techniques [16] and loop MHV methods [17] are successful in obtaining the cut-constructible parts of amplitudes; essentially this is because at some level they are dealing with four-dimensional cuts. In principle performing D-dimensional cuts generates all parts of amplitudes [18 − 21] as long as only massless particles are involved, however these techniques still appear to be relatively cumbersome. Combinations of recursive techniques and unitarity have led to important progress recently [22 − 31], but it would be preferable to have a more powerful prescriptive formulation, particularly keeping in mind that applications to more general situations are sought.

A promising development from this point of view is the Lagrangian approach [32 − 34]. Here it has been argued that lightcone Yang-Mills theory, combined with a certain change of field variables, yields a classical action which comprises precisely the MHV vertices. A full Lagrangian description of MHV techniques would in principle give a prescription for applying such methods to diverse theories. The next step in developing this is to understand the quantum corrections in this Lagrangian approach. If one directly uses in a path integral the classical MHV action, containing only purely four-dimensional MHV vertices, then it is immediately clear that this cannot yield all known quantum amplitudes. For example, there is no way to construct one-loop amplitudes where the external gluons all have positive helicities, or where only one gluon has negative helicity, as all MHV vertices contain two negative helicity particles (this issue has been recently discussed in [35]). These amplitudes are particular cases where the entire amplitude consists of rational terms. More generally, it seems clear that the vertices of the classical MHV Lagrangian will not yield the rational parts of amplitudes, but only the cut-constructible terms [15]. Important insights into this question can be obtained from the study of self-dual Yang-Mills theory, which has the same all-plus one-loop amplitude of full YM [36 − 38] as its sole quantum correction.[1] An example, relevant to the discussion in this paper, is given in [35] where it was shown how these amplitudes might be obtained from the Jacobian arising from a Bäcklund-type change of variables which takes the self-dual Yang-Mills theory to a free theory.

A discussion of the full Yang-Mills theory in the lightcone gauge has recently been given by Chakrabarti, Qiu and Thorn (CQT) in [39 − 41]. These papers employ an interesting regularisation which, importantly, does not change the dimension of spacetime. For this reason, we find it particularly suitable for setting the scene for the MHV diagram method, which is inherently four-dimensional in current approaches. The regularisation of CQT requires the introduction of certain counterterms, which prove to be rather simple in form. What we will show in this paper is that these simple counterterms provide a very compact and powerful way to represent the rational terms in gauge theory amplitudes; specifically, we will demonstrate that the single two-point counterterm contains *all* the n-point all-plus amplitudes. The way this happens is through the use of the new field variables of [32 − 34]. Other counterterms will combine with vertices from the Lagrangian and should generate the rational parts of more general amplitudes. Based on the discussion in this paper, we propose that the counterterms, expressed in the field variables which give rise to standard MHV vertices, in combination with Lagrangian vertices, generate the rational terms previously missing from MHV diagram formulations.

The rest of the paper is organised as follows. After giving some background material in section 2, we explicitly derive in section 3 the four point all-plus amplitude from the two-point counterterm of CQT. We follow this by showing that the n-point expression, obtained by writing the counterterm in new variables, has precisely the right collinear and soft limits required for it to be the correct all-plus n-point amplitude. We present our conclusions in section 4, and our notation and derivations of certain identities have been collected in two appendices.

[1] In real Minkowski space, this is in fact its single non-vanishing amplitude.

JHEP07(2007)002

2. Background

In this section, we first review the classical field redefinition from the lightcone Yang-Mills Lagrangian to the MHV-rules Lagrangian. We then move on to motivate the four-dimensional regularisation scheme we will employ, and argue that it leads directly to the introduction of a certain Lorentz-violating counterterm in the Yang-Mills Lagrangian. We close the section with the remarkable observation that this counterterm provides a simple way to calculate the four-point all-plus one-loop amplitude using only tree-level combinatorics.

2.1 The classical MHV lagrangian

It seemed clear from the beginning that the MHV diagram approach to Yang-Mills theory must be closely related to lightcone gauge theory. This idea was substantiated by Mansfield [33] (see also [32]). The starting point of [33] is the lightcone gauge-fixed YM Lagrangian for the fields corresponding to the two physical polarisations of the gluon. It was argued convincingly in [33] that a certain canonical change of the field variables re-expresses this lightcone Lagrangian as a theory containing the infinite series of MHV vertices. Some of the arguments in [33] were rather general; these were reviewed in [34], where the change of variables was discussed in more detail, and in particular it was shown how the four- and five-point MHV vertices arise from the change of variables. In this paper we will mainly follow the notation of [34].

The general structure of the lightcone YM Lagrangian, after integrating out unphysical degrees of freedom, is (see appendix A for more details)

$$\mathcal{L}_{\rm YM} = \mathcal{L}_{+-} + \mathcal{L}_{++-} + \mathcal{L}_{--+} + \mathcal{L}_{++--}\,, \tag{2.1}$$

where the gauge condition is $\eta^\mu A_\mu = 0$ with the null vector $\eta = (1/\sqrt{2}, 0, 0, 1/\sqrt{2})$. Since this Lagrangian contains a $++-$ vertex, it is not of MHV type. In [33], Mansfield proposed to eliminate this vertex through a suitably chosen field redefinition. Specifically, he performed a canonical change of variables from $(A, \bar{A})$ to new fields $(B, \bar{B})$, in such a way that

$$\mathcal{L}_{+-}(A, \bar{A}) + \mathcal{L}_{++-}(A, \bar{A}) = \mathcal{L}_{+-}(B, \bar{B})\ . \tag{2.2}$$

The remarkable result is that upon inserting this change of variables into the remaining two vertices, the Lagrangian, written in terms of $(B, \bar{B})$, becomes a sum of MHV vertices,

$$\mathcal{L}_{\rm YM} = \mathcal{L}_{+-} + \mathcal{L}_{+--} + \mathcal{L}_{++--} + \mathcal{L}_{+++--} + \dots\ . \tag{2.3}$$

The crucial property of Mansfield's transformation that makes this possible is that, while both A and $\bar{A}$ are series expansions in the new B fields, A has no dependence on the $\bar{B}$ fields while $\bar{A}$ turns out to be *linear* in $\bar{B}$. Thus, since the remaining vertices are quadratic in the $\bar{B}$, the new interaction vertices have the helicity configuration of an MHV amplitude. Mansfield was also able to show that the explicit form of the vertices coincides with the CSW off-shell continuation of the Parke-Taylor formula for the MHV scattering amplitudes, as proposed by [7].

JHEP07(2007)002

One of the main results of [34] was the derivation of an explicit, closed formula for the expansion of the original fields $(A, \bar{A})$ in terms of the new fields $(B, \bar{B})$. This was then used to verify that the new vertices are indeed precisely the MHV vertices of [7], at least up to the five-point level. We will now briefly review these results. First, recall that the positive helicity field A is a function of the positive helicity B field only. It is expanded as follows:

$$A(\vec{p}) = \sum_{n=1}^{\infty} \int_{\Sigma} \prod_{i=1}^{n} \frac{\mathrm{d}^3 p^i}{(2\pi)^3}\, \Delta(\vec{p}, \vec{p}^1, \ldots \vec{p}^n)\; \mathrm{Y}(\vec{p}; 1 \cdots n)\; B(\vec{p}^1) B(\vec{p}^2) \cdots B(\vec{p}^n)\,, \tag{2.4}$$

where $\Delta(\vec{p}, \vec{p}^1, \ldots \vec{p}^n) := (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}^1 - \cdots - \vec{p}^n)$. Note that the x^- coordinate is common to all the fields, which is why we have restricted the transformation to the lightcone quantisation surface Σ.

By inserting this expansion into (2.2) and using the requirement that the transformation be canonical, Ettle and Morris succeeded in deriving a very simple expression for the coefficients Y. After translating to our conventions (see appendix A), they are given by:

$$\mathrm{Y}(\vec{p}; 12 \cdots n) = (\sqrt{2} i g)^{n-1} \frac{p_+}{\sqrt{p_+^1 p_+^n}}\; \frac{1}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n-1, n \rangle}\,. \tag{2.5}$$

The first few terms in (2.4) are then:

$$\begin{aligned} A(\vec{p}) \;=\; & B(\vec{p}) + \sqrt{2} i g p_+ \int_{\Sigma} \frac{\mathrm{d}^3 p^1 \mathrm{d}^3 p^2}{(2\pi)^3} \frac{\delta^{(3)}(\vec{p} - \vec{p}^1 - \vec{p}^2)}{\sqrt{p_+^1 p_+^2}} \frac{1}{\langle 12 \rangle}\; B(\vec{p}^1) B(\vec{p}^2) \\ & - 2 g^2 p_+ \int_{\Sigma} \frac{\mathrm{d}^3 p^1 \mathrm{d}^3 p^2 \mathrm{d}^3 p^3}{(2\pi)^6} \frac{\delta^{(3)}(\vec{p} - \vec{p}^1 - \vec{p}^2 - \vec{p}^3)}{\sqrt{p_+^1 p_+^3}} \frac{1}{\langle 12 \rangle \langle 23 \rangle} B(\vec{p}^1) B(\vec{p}^2) B(\vec{p}^3) + \cdots\,. \end{aligned} \tag{2.6}$$

Similarly, one can write down the expansion of the negative helicity field $\bar{A}$, which, as discussed above, is linear in $\bar{B}$, but is an infinite series in the new field B. In [34] it was shown that the coefficients in the expansion of $\bar{A}$ are very closely related to those for A.[2] The expansion of $\bar{B}$ turns out to be simply

$$\begin{aligned} \bar{A}(\vec{p}) \;=\; & -\sum_{n=1}^{\infty} \sum_{s=1}^{n} \int_{\Sigma} \prod_{i=1}^{n} \frac{\mathrm{d}^3 p^i}{(2\pi)^3} \Delta(\vec{p}, \vec{p}^1, \ldots, \vec{p}^n) \frac{(p_+^s)^2}{(p_+)^2}\; \mathrm{Y}(\vec{p}; 1 \cdots n) B(\vec{p}^1) \cdots \bar{B}(\vec{p}_s) \cdots B(\vec{p}^n) \\ =\; & -\sum_{n=1}^{\infty} \int_{\Sigma} \prod_{i=1}^{n} \frac{\mathrm{d}^3 p^i}{(2\pi)^3}\; \Delta(\vec{p}, \vec{p}^1, \ldots, \vec{p}^n)\; \frac{1}{(p_+)^2} \mathrm{Y}(\vec{p}; 1 \cdots n) \\ & \qquad \times \sum_{s=1}^{n} (p_+^s)^2\; B(\vec{p}^1) \cdots \bar{B}(\vec{p}^s) \cdots B(\vec{p}^n). \end{aligned} \tag{2.7}$$

[2] This is perhaps easiest to see [42] by considering that, in the context of $\mathcal{N}=4$ SYM, A and B are part of the same lightcone superfield.

Thus we see that at each order in the expansion, we need to sum over all possible positions of $\bar{B}$. Explicitly, the first few terms are:

$$
\begin{aligned}
\bar{A}(\vec{p})= & \bar{B}(\vec{p}) - \sqrt{2}ig \int_\Sigma \frac{\mathrm{d}^3p^1\mathrm{d}^3p^2}{(2\pi)^3}\delta^{(3)}(\vec{p}-\vec{p}^{\,1}-\vec{p}^{\,2})\frac{1}{p_+\sqrt{p_+^1p_+^2}}\frac{1}{\langle 12\rangle}\times \\
& \times\left[(p_+^1)^2\bar{B}(\vec{p}^{\,1})B(\vec{p}^{\,2})+(p_+^2)^2B(\vec{p}^{\,1})\bar{B}(\vec{p}^{\,2})\right] \\
& +2g^2\int_\Sigma \frac{\mathrm{d}^3p^1\mathrm{d}^3p^2\mathrm{d}^3p^3}{(2\pi)^6}\delta^{(3)}(\vec{p}-\vec{p}^{\,1}-\vec{p}^{\,2}-\vec{p}^{\,3})\frac{1}{p_+\sqrt{p_+^1p_+^3}}\frac{1}{\langle 12\rangle\langle 23\rangle}\times \\
& \times[(p_+^1)^2\bar{B}(\vec{p}^{\,1})B(\vec{p}^{\,2})B(\vec{p}^{\,3})+(p_+^2)^2B(\vec{p}^{\,1})\bar{B}(\vec{p}^{\,2})B(\vec{p}^{\,3})+(p_+^3)^2B(\vec{p}^{\,1})B(\vec{p}^{\,2})\bar{B}(\vec{p}^{\,3})]+\cdots
\end{aligned} \tag{2.8}
$$

Using the above results, it is in principle straightforward to derive the terms that arise on inserting the Mansfield transformation into the two remaining vertices of the theory. For the simplest cases, one can see explicitly that these combine to produce MHV vertices, and some arguments were also given in [33, 34] that this must be true in general.

In supersymmetric theories, the MHV vertices are enough to reproduce complete scattering amplitudes at one loop [43]. However, as we mentioned earlier, for pure YM it is clear that the terms in the MHV Lagrangian (2.3) will not be enough to generate complete quantum amplitudes. For instance, the scattering amplitude with all gluons with positive helicity, which at one loop is finite and given by a rational term, cannot be obtained by only using MHV diagrams, for the simple reason that one cannot draw any diagram contributing to it by only resorting to MHV vertices.[3] Another amplitude which cannot be derived within conventional MHV diagrams is the amplitude with only one gluon of negative helicity. Similarly to the all-plus amplitude, this single-minus amplitude vanishes at tree level, and at one loop is given by a finite, rational function of the spinor variables.

The lesson we learn from this is that, in order to apply the MHV method to derive complete amplitudes in pure YM, one should look more closely at the change of variables in the full quantum theory. There are several possible subtleties one should pay careful attention to at the quantum level. First of all, it is possible that the canonical nature of the transformation is not preserved, leading to a non-trivial Jacobian which could provide the missing amplitudes. Another possible source of contributions could come from violations of the equivalence theorem. This theorem states that, although correlation functions of the new fields are in general different from those of the old fields, the scattering amplitudes are actually the same,[4] as long as the new fields are good interpolating fields. These issues were explored in some detail in [35] (see also [34, 42]) where it was shown, for a different (non-canonical) field redefinition, how a careful treatment of these effects can combine to reproduce some of the amplitudes that would seem to be missing at first sight.

Another aim of [35] was to demonstrate how to reproduce one of the above-mentioned rational amplitudes, the one with all-minus helicities, in the MHV formalism. This amplitude is slightly less mysterious than the all-plus amplitude in the sense that one can write

[3]On the other hand, it was shown in [35] that the parity conjugate all-minus amplitude is correctly generated by using MHV diagrams.

[4]Modulo a trivial wave-function renormalisation.

down the contributing diagrams using only MHV vertices; however a calculation without a suitable regulator in place would give a vanishing answer, despite the fact that this amplitude is finite. In [35], it was shown, using dimensional regularisation, that the full nonzero result arises from a slight mismatch between four– and D $(= 4-2\epsilon)$–dimensional momenta.

It is natural therefore to expect that dimensional regularisation will be helpful also for the problem at hand, which is to recover the rational amplitudes of pure Yang-Mills after the Mansfield transformation. Decomposing the regularised lightcone Lagrangian into a pure four-dimensional part and the remaining ϵ-dependent terms, and performing the transformation on the four-dimensional part only, will give rise to several new ϵ-dependent terms that can potentially give finite answers when forming loops.

Although this approach shows promise, it is not the one we will make use of in the following. Instead, motivated by the fact that the Mansfield transformation seems to be deeply rooted in four dimensions, we would like to look for a purely four-dimensional regularisation scheme. We now turn to a review of the particular scheme we will use.

2.2 A four-dimensional regulator for lightcone Yang-Mills

In the above we explained why a naïve application of the Mansfield transform leads to puzzles at the quantum level, and discussed possible ways to improve the situation. The conclusion was that, since the missing amplitudes arise from subtle mismatches in regularisation, one should be careful to perform the Mansfield transform on a suitably regularised version of the lightcone Yang-Mills action. Here we will review one approach to the regularisation of lightcone Yang-Mills, which, despite several slightly unusual features, appears to be ideally suited for the problem at hand.

The regularisation we propose to use is inspired by recent work of CQT [39 – 41] on Yang-Mills amplitudes in the lightcone worldsheet approach [44, 45]. This is an attempt to understand gauge-string duality which is similar in spirit to 't Hooft's original work on the planar limit of gauge theory [46], and aims at improving on early dual model techniques [47, 48]. We recall that one of the main goals in those works is to exhibit the string worldsheet as made up of very large planar diagrams ("fishnets").

In their recent work, Thorn and collaborators make this statement more precise, using techniques that were unavailable when the original ideas were put forward. It is hoped that, by understanding how to translate a generic Yang-Mills planar diagram to a configuration of fields (with suitable boundary conditions) on the lightcone worldsheet, it will eventually become possible to perform the sum of all these diagrams. This approach to gauge-string duality is thus complementary to that using the AdS/CFT correspondence.

The field content and structure of the worldsheet theory dual to Yang-Mills theory is rather intricate (see e.g. [45]), but for our purposes the details are not important. What is most relevant for us is that one of the principles of this approach is that all quantities on the Yang-Mills side should have a local worldsheet description. This includes the choice of regulator that needs to be introduced when calculating loop diagrams. This requirement led Thorn [49] (see also [50, 51]) to introduce an exponential UV cutoff, which we will discuss in a short while.

JHEP07(2007)002

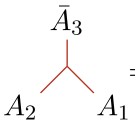
Since one of the goals of this programme is to translate an arbitrary planar diagram into worldsheet form (and eventually calculate it), it is an important intermediate goal to understand how to do standard Yang-Mills perturbation theory in “worldsheet-friendly” language. In [39 – 41] CQT do exactly that for the simplest case, that of one-loop diagrams in Yang-Mills theory, by analysing how familiar features like renormalisation are affected by the unusual regularisation procedure and other special features of the lightcone worldsheet formalism.

To conclude this brief overview of the lightcone worldsheet formalism, the main point for our current purposes is that it provides motivation and justification for a slightly unusual regularisation of lightcone Yang-Mills, which we will now describe.

Let us momentarily focus on the self-dual part of the lightcone Yang-Mills Lagrangian:

$$\mathcal{L} = \mathcal{L}_{-+} + \mathcal{L}_{++-} = -A_{\bar{z}} \Box A_z + 2ig[A_z, \partial_+ A_{\bar{z}}](\partial_+)^{-1}(\partial_{\bar{z}} A_z) \ . \tag{2.9}$$

This action provides one of the representations of self-dual Yang-Mills theory. After transforming to momentum space, we find that the only vertex in the theory is the following (suppressing the gauge index structure):



$$= -2g \frac{p_+^3}{p_+^1 p_+^2} [p_+^1 p_{\bar{z}}^2 - p_+^2 p_{\bar{z}}^1] = -\sqrt{2} g \frac{p_+^3}{\sqrt{p_+^1 p_+^2}} [12] \ . \tag{2.10}$$

As for propagators, following [40], we will use the Schwinger representation:

$$\frac{1}{p^2} = -\int_0^\infty \mathrm{d}T e^{+Tp^2} \ . \tag{2.11}$$

In (2.11) p^2 is understood to be the appropriate ($p^2 < 0$) Wick rotated version of the Minkowski space inner product. For our choice of signature, the latter is

$$p \cdot q = p_+ q_- + p_- q_+ - \mathbf{p} \cdot \mathbf{q} = p_+ q_- + p_- q_+ - (p_z q_{\bar{z}} + p_{\bar{z}} q_z) \,, \tag{2.12}$$

so that $p^2 = 2(p_+ p_- - p_z p_{\bar{z}})$.

We will also make use of the dual or “region momentum” representation, where one assigns a momentum to each *region* that is bounded by a line in the planar diagram. By convention, the actual momentum of the line is given by the region momentum to its right minus that on its left, as given by the direction of momentum flow.[5] Clearly such a prescription can only be straightforwardly implemented for planar diagrams, which is the case considered in [40]. This is also sufficient for our purposes, since we are calculating the leading single-trace contribution to one-loop scattering amplitudes. Non-planar (multi-trace) contributions can be recovered from suitable permutations of the leading-trace ones (see e.g. [52]).

To demonstrate the use of region momenta, a sample one-loop diagram is pictured in figure 1.

[5] In [40] the flow of momentum is chosen to always match the flow of helicity, but we will not use this convention.

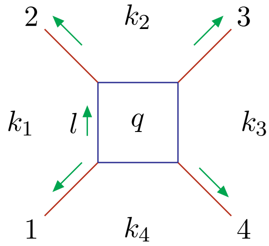


Figure 1: A sample one-loop diagram indicating the labelling of region momenta. The outgoing leg momenta are $p_1 = k_1 - k_4$, $p_2 = k_2 - k_1$, $p_3 = k_3 - k_2$, $p_4 = k_4 - k_3$, while the loop momentum (directed as indicated) is $l = q - k_1$.

The "worldsheet-friendly" regulator that CQT employ is simply defined as follows [49]: For a general n-loop diagram, with q_i being the loop region momenta, one simply inserts an exponential cutoff factor

$$\exp\left(- \delta \sum_{i=1}^{n} \mathbf{q}_i^2 \right) \tag{2.13}$$

in the loop integrand, where δ is positive and will be taken to zero at the end of the calculation. This clearly regulates UV divergences (from large transverse momenta), but, as we will see, has some surprising consequences since it will lead to finite values for certain Lorentz-violating processes, which therefore have to be cancelled by the introduction of appropriate counterterms.

Note that $\mathbf{q}^2 = 2q_z q_{\bar{z}}$ has components only along the two transverse directions, hence it breaks explicitly even more Lorentz invariance than the lightcone usually does. This might seem rather unnatural from a field-theoretical point of view, however it is crucial in the lightcone worldsheet approach. Indeed, the lightcone time x^- and x^+ (or in practice its dual momentum p_+) parametrise the worldsheet itself, and are regulated by discretisation; thus, they are necessarily treated very differently from the two transverse momenta $q_z, q_{\bar{z}}$ which appear as dynamical worldsheet scalars. Fundamentally, this is because of the need to preserve longitudinal (x^+) boost invariance (which eventually leads to conservation of discrete p_+). The fact that the regulator depends on the region momenta rather than the actual ones is a consequence of asking for it to have a local description on the worldsheet.

The main ingredient for what will follow later in this paper is the computation of the (++) one-loop gluon self-energy in the regularisation scheme discussed earlier. This is performed on page 10 of [40], and we will briefly outline it here. This helicity-flipping gluon self-energy, which we denote by Π^{++}, is the only potential self-energy contribution in self-dual Yang-Mills; in full YM we would also have Π^{+-} and, by parity invariance, Π^{--}.

There are two contributions to this process, corresponding to the two ways to route helicity in the loop, but they can be easily shown to be equal so we will concentrate on one of them, which is pictured in figure 2.

In figure 2, $p, -p$ are the outgoing line momenta, l is the loop line momentum, and

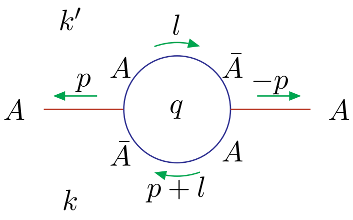


Figure 2: Labelling of one of the selfenergy diagrams contributing to Π^{++}.

k, k', q are the region momenta, in terms of which the line momenta are given by

$$p = k' - k, \quad l = q - k' \ . \tag{2.14}$$

Remembering to double the result of this diagram, we find the following expression for the self-energy:

$$\begin{aligned}
\Pi^{++} &= 8g^2 N \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \left[\frac{-(p+l)_+}{p_+ l_+} (p_+ l_{\bar{z}} - l_+ p_{\bar{z}}) \right] \times \frac{1}{l^2 (p+l)^2} \times \\
&\qquad \times \left[\frac{-l_+}{(-p_+)(p+l)_+} ((-p_+)(p_{\bar{z}} + l_{\bar{z}}) - (p_+ + l_+)(p_{\bar{z}})) \right] \\
&= \frac{g^2 N}{2\pi^4} \int \mathrm{d}^4 l \frac{1}{(p_+)^2} (p_+ l_{\bar{z}} - l_+ p_{\bar{z}})(p_+(p_{\bar{z}} + l_{\bar{z}}) - (p_+ + l_+) p_{\bar{z}}) \frac{1}{l^2 (p+l)^2} .
\end{aligned} \tag{2.15}$$

Although we are suppressing the colour structure, the factor of N is easy to see by thinking of the double-line representation of this diagram.[6] One of the crucial properties of (2.15) is that the factors of the loop momentum l_+ coming from the vertices have cancelled out, hence there are no potential subtleties in the loop integration as $l_+ \to 0$. This means that, although for general loop calculations one would have to follow the DLCQ procedure and discretise l_+ (as is done for other processes considered in [39 – 41]), this issue does not arise at all for this particular integral, and we are free to keep l_+ continuous.

To proceed, we convert momenta to region momenta, rewrite propagators in Schwinger representation, and regulate divergences using the regulator (2.13). Employing the unbroken shift symmetry in the + region momenta to further set $k_+ = 0$, (2.15) can be recast as:

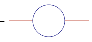
$$\begin{aligned}
\Pi^{++} = \frac{g^2 N}{2\pi^4} &\int_0^\infty \mathrm{d}T_1 \mathrm{d}T_2 \int \mathrm{d}^4 q \frac{1}{(k'_+)^2} e^{T_1 (q-k)^2 + T_2 (q-k')^2 - \delta \mathbf{q}^2} \times \\
&\times \left[k'_+ (q_{\bar{z}} - k'_{\bar{z}}) - (q_+ - k'_+)(k'_{\bar{z}} - k_{\bar{z}}) \right] \left[k'_+ (q_{\bar{z}} - k_{\bar{z}}) - q_+ (k'_{\bar{z}} - k_{\bar{z}}) \right] \ .
\end{aligned} \tag{2.16}$$

Since q_- only appears in the exponential, the q_- integration will lead to a delta function containing q_+, which can be easily integrated and leads to a Gaussian-type integral for $q_z, q_{\bar{z}}$. Performing this integral, we obtain (setting $T = T_1 + T_2$, $x = T_1/(T_1 + T_2)$)

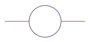
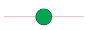
$$\Pi^{++} = \frac{g^2 N}{2\pi^2} \int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}T \ \delta^2 \ \frac{[x k_{\bar{z}} + (1-x) k'_{\bar{z}}]^2}{(T+\delta)^3} \ e^{T x (1-x) p^2 - \frac{\delta T}{T+\delta} (x \mathbf{k} + (1-x) \mathbf{k}')^2} \ . \tag{2.17}$$

[6] For simplicity, we take the gauge group to be U(N).

Notice that, had we not regularised using the δ regulator, we would have obtained zero at this stage. Instead, now we can see that there is a region of the T integration (where $T \sim \delta$) that can lead to a nonzero result. On performing the T and x integrations, and sending δ to zero at the end, we obtain the following finite answer:

$\Pi^{++} = 2\Big(+$  $+\Big) = \frac{g^2 N}{12\pi^2}\left((k_{\bar{z}})^2 + (k'_{\bar{z}})^2 + k_{\bar{z}} k'_{\bar{z}}\right)\,.$ (2.18)

Notice that this nonvanishing, finite result violates Lorentz invariance, since it would imply that a single gluon can flip its helicity. Also, it explicitly depends on only the $\bar{z}$ components of the region momenta. Such a term is clearly absent in the tree-level Lagrangian (unlike e.g. the Π^{+-} contribution in full Yang-Mills theory), thus it cannot be absorbed through renormalisation — it will have to be explicitly cancelled by a counterterm. This counterterm, which will play a major rôle in the following, is defined in such a way that:

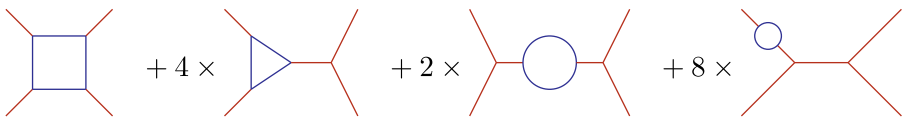
 $+$  $= 0\,,$ (2.19)

in other words it will cancel all insertions of Π^{++}, diagram by diagram. Let us note here that, had we been doing dimensional regularisation, all bubble contributions would vanish on their own, so there would be no need to add any counterterms. So this effect is purely due to the "worldsheet-friendly" regulator (2.13).

It is also interesting to observe that in a supersymmetric theory this bubble contribution would vanish[7] so this effect is only of relevance to pure Yang-Mills theory.

2.3 The one-loop (++++) amplitude

Now let us look at the all-plus four-point one-loop amplitude in this theory. It is easy to see that it will receive contributions from three types of geometries: boxes, triangles and bubbles. It is a remarkable property[8] that the sum of all these geometries adds up to zero. In particular, with a suitable routing of momenta, the integrand itself is zero. Pictorially, we can state this as:

 $= 0\,.$ (2.20)

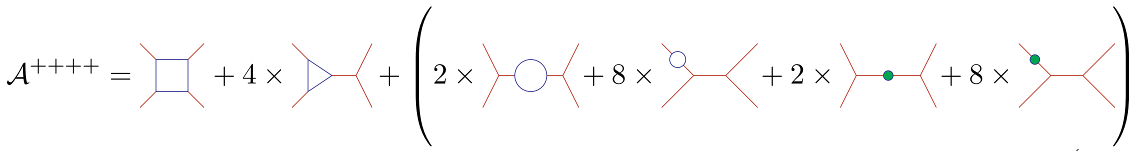
The coefficients mean that we need to add that number of inequivalent orderings. So we see (and refer to [40] for the explicit calculation) that the sum of all the diagrams that one can construct from the single vertex in our theory, gives a vanishing answer. However, as discussed in the previous section, this is not everything: we need to also include the contribution of the counterterm that we are forced to add in order to preserve Lorentz invariance. Since this counterterm, by design, cancels all the bubble graph contributions,

[7] This can in fact be derived from the results of [53], where similar calculations were considered with fermions and scalars in the loop.

[8] This observation is attributed to Zvi Bern [40].

JHEP07(2007)002

we are left with just the sum of the box and the four triangle diagrams. In pictures,



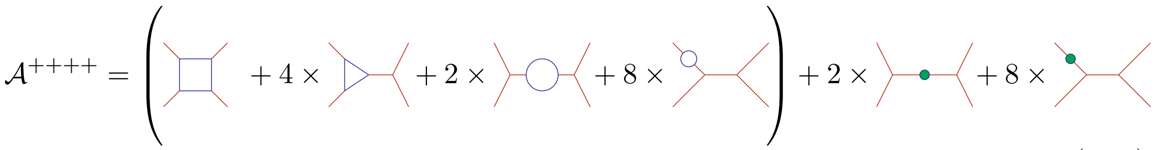
(2.21)

where $\mathcal{A}^{++++}$ is the known result [54] for the leading-trace part of the four-point all-plus amplitude:

$$\mathcal{A}^{++++}(A_1 A_2 A_3 A_4) = i \frac{g^4 N}{48\pi^2} \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\,, \tag{2.22}$$

and the terms in the parentheses clearly cancel among themselves. This leaves the box and triangle diagrams, which are exactly those appearing in the calculation of the parity conjugate amplitude using dimensional regularisation [35], where the bubbles were zero to begin with.

Following [40], we make the obvious, but important for the following, observation that one can change the position of the parentheses:



(2.23)

where again the terms in the parentheses are zero (by (2.20)). This demonstrates that one can compute the all-plus amplitude just from a tree-level calculation with counterterm insertions (of course, these diagrams are at the same order of the coupling constant as one-loop diagrams because of the counterterm insertion). This remarkable claim is verified in [40], where CQT explicitly calculate the 10 counterterm diagrams and recover the correct result for the four-point amplitude (see pp. 22-23 of [40]).[9]

This result, apart from being very appealing in that one does not have to perform any integrals (apart from the original integral that defined the counterterm) so that the calculation reduces to tree-level combinatorics, will also turn out to be a convenient starting point for performing the Mansfield transformation. Specifically, our claim is that the whole series of all-plus amplitudes will arise just from the counterterm action. In the following we will show how this works explicitly for the four-point all-plus case, and then we will argue for the n-point case that the corresponding expression derived from the counterterm has all the correct singularities (soft and collinear), giving strong evidence that the result is true in general.

3. The all-plus amplitudes from a counterterm

Having reviewed the relevant new features that arise when doing perturbation theory with the worldsheet-motivated regulator of [49], we now have all the necessary ingredients to

[9] In practice, these authors choose to insert the self-energy result (2.18) in the tree diagrams, so what they compute is *minus* the all-plus amplitude.

perform the Mansfield change of variables on the regulated lightcone Lagrangian. In this section, we will carry out this procedure. We will first regulate lightcone self-dual Yang-Mills, which, as discussed, will require us to introduce an explicit counterterm in the Lagrangian. Then we will perform the Mansfield transformation on the original Lagrangian (converting it to a free theory). We will then show that, upon inserting the change of variables into the counterterm Lagrangian, we recover the all-plus amplitudes as vertices in the theory.

3.1 Mansfield transformation of $\mathcal{L}_{\mathrm{CT}}$

As we saw, the "worldsheet-friendly" regularisation requires us to add a certain counterterm to the lightcone Yang-Mills action, required in order to cancel the Lorentz-violating helicity-flipping gluon selfenergy. As mentioned previously, the calculation of the all-plus amplitude can be tackled purely within the context of *self-dual* Yang-Mills, which we will focus on from now on. We see that, as a result of this regularisation, the complete action at the quantum level becomes:

$$\mathcal{L}^{(r)}_{\mathrm{SDYM}} = \mathcal{L}_{+-} + \mathcal{L}_{++-} + \mathcal{L}_{\mathrm{CT}}\,, \tag{3.1}$$

where $\mathcal{L}_{+-} + \mathcal{L}_{++-}$ is the classical Lagrangian for self-dual Yang-Mills introduced in (2.9). Although CQT do not write down a spacetime Lagrangian for $\mathcal{L}_{\mathrm{CT}}$, it is easy to see that the following expression would have the right structure:

$$\mathcal{L}_{\mathrm{CT}} = -\frac{g^2 N}{12\pi^2}\int_\Sigma \mathrm{d}^3k^i \mathrm{d}^3k^j \; {A^i}_j(k^i,k^j)[(k^i_{\bar z})^2 + (k^j_{\bar z})^2 + k^i_{\bar z}k^j_{\bar z}]{A^j}_i(k^j,k^i)\;. \tag{3.2}$$

This expression depends explicitly on the dual, or region, momenta. In (3.2) we have made use of the simplest way to associate region momenta to fields, which is to assign a region momentum to each *index* line in double-line notation [46], and thus a momentum k^i, k^j to each of the indices of the gauge field ${A^i}_j$ (now slightly extended into a dipole, as would be natural from the worldsheet perspective, where an index is associated to each boundary). Since each line has a natural orientation, the actual momentum of each line can be taken to be the difference of the index momentum of the incoming index line and the outgoing index line. So the momentum of ${A^i}_j(k^i,k^j)$ is taken to be $p = k^j - k^i$. As discussed above, this assignment can only be performed consistently for planar diagrams, which is sufficient for our purposes.

Clearly, the structure of (3.2) is rather unusual. First of all, it depends only on the antiholomorphic ($\bar z$) components of the region momenta, and so is clearly not (lightcone) covariant. Even more troubling is the fact that it does not depend only on *differences* of region momenta, but also on their sums. Since each field thus carries more information than just its momentum, $\mathcal{L}_{\mathrm{CT}}$ is a non-local object from a four-dimensional point of view (although, as shown in [40], it can be given a perfectly local worldsheet description).

Leaving the above discussion as food for thought, we will now rewrite (3.2) in a more conventional way that is most convenient for inserting into Feynman diagrams,

$$\mathcal{L}_{\mathrm{CT}} = -\frac{g^2 N}{12\pi^2}\int_\Sigma \mathrm{d}^3p\,\mathrm{d}^3p'\,\delta(p+p')\; {A^i}_j(p')((k^i_{\bar z})^2 + (k^j_{\bar z})^2 + k^i_{\bar z}k^j_{\bar z}){A^j}_i(p)\;. \tag{3.3}$$

In this expression, which can be thought of as the zero-mode or field theory limit of (3.2), all the region momentum dependence is confined to the polynomial factor $(k^i_{\bar{z}})^2+(k^j_{\bar{z}})^2+k^i_{\bar{z}}k^j_{\bar{z}}$. This vertex, inserted into tree diagrams, would exactly reproduce the effects of the counterterm pictured in (2.19). Although (3.3) still exhibits some of the apparently undesirable features we discussed above, the calculations in [40] demonstrate that, after summing over all possible insertions of this term, the final result is covariant and correctly reproduces the all-plus amplitudes.[10] Therefore, we believe that its problematic properties are really a virtue in disguise, and (as we will see explicitly) they seem to be crucial in obtaining the full series of n-point all-plus amplitudes from the Mansfield transformation of a *single* term.

We are now ready to perform the Mansfield change of variables. In the spirit of the discussion earlier, we will perform the transformation on the *classical* part of the action only:

$$\mathcal{L}_{+-}(A,\bar{A})+\mathcal{L}_{++-}(A,\bar{A})=\mathcal{L}_{+-}(B,\bar{B}) \tag{3.4}$$

Hence the classical part of the action has been converted to a free theory. Without a regulator, this would be the whole story. However we now see that, within the particular regularisation we are working with, the full Lagrangian $\mathcal{L}^{(r)}_{\rm SDYM}$ contains one extra, one-loop piece, given by $\mathcal{L}_{\rm CT}$ in (3.3), which is quadratic in the positive helicity fields A. To complete the Mansfield transformation, we will clearly need to expand this term in the new fields B, using the Ettle-Morris coefficients (2.4).

Since $\mathcal{L}_{\rm CT}$ depends only on the holomorphic A fields, we will only need the expansion of A in terms of B given in (2.4). As a first check that $\mathcal{L}_{\rm CT}$ leads to the right kind of structure, note that since A depends only on the holomorphic B fields, all the new vertices are all-plus. Thus, the full action, when expressed in terms of the B fields, takes the schematic form:

$$\mathcal{L}^{(r)}_{\rm SDYM}(A,\bar{A})=\mathcal{L}_{+-}(B,\bar{B})+\mathcal{L}_{++}(B)+\mathcal{L}_{+++}(B)+\mathcal{L}_{++++}(B)+\cdots \tag{3.5}$$

In the next section we will calculate the four-point term $\mathcal{L}_{++++}$ and demonstrate that, when restricted on-shell, it reproduces the known form (2.22) for the all-plus amplitude.

3.2 The four-point case

To begin with, we focus on the derivation of the four-point all-plus vertex, whose on-shell version will give us the four-point scattering amplitude. We will thus expand the old fields A in the counterterm (3.3) (or (3.2)) up to terms containing four B-fields.

When inserting the Ettle-Morris coefficients into (3.3), one has to sum over all possible cyclic orderings with which this can be done. A complication is that now the counterterm itself depends on the ordering. In other words, we need to sum over all the ways of assigning

[10] Note that similar-looking treatments using index momenta instead of line momenta for vertices, but which in the end sum up to covariant results have appeared in the context of noncommutative geometry (see e.g. [55]). Although it is possible to write e.g. (3.2) in star-product form, at this stage it is not clear whether that is a useful reformulation.

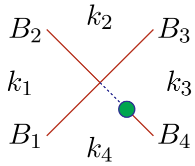


Figure 3: One of the contributions to the four-point all-plus vertex.

dual momenta to the indices. Schematically, the inequivalent terms that we obtain are:

$$\begin{aligned} AA \to & (\textstyle\int B_1B_2)(\textstyle\int B_3B_4) + (\textstyle\int B_2B_3)(\textstyle\int B_4B_1) \\ & + (\textstyle\int B_1B_2B_3)B_4 + (\textstyle\int B_2B_3B_4)B_1 + (\textstyle\int B_3B_4B_1)B_2 + (\textstyle\int B_4B_1B_2)B_3\,, \end{aligned} \tag{3.6}$$

where the terms on the first line arise from doing two quadratic substitutions and those on the second from doing one cubic substitution. All the other possibilities are related by cyclicity of the trace. For definiteness, let us now write down what one of these terms means explicitly:[11]

$$\begin{aligned} \left(\int B_1B_2B_3\right)B_4 = & -2g^2 \mathrm{tr}\int \mathrm{d}p\mathrm{d}p^4\delta(p+p^4)\left[\int \mathrm{d}p^1\mathrm{d}p^2\mathrm{d}p^3\delta(p-p^1-p^2-p^3)\frac{p_+}{\sqrt{p^1_+p^3_+}}\frac{1}{\langle 12\rangle\langle 23\rangle}\times\right. \\ & \left.\times B(p^1)B(p^2)B(p^3)\right]\left[(k^3_{\bar z})^2+(k^4_{\bar z})^2+k^4_{\bar z}k^3_{\bar z}\right]\ B(p^4) \\ = & 2g^2 \int \mathrm{d}p^1\mathrm{d}p^2\mathrm{d}p^3\mathrm{d}p^4\delta(p^1+p^2+p^3+p^4)\times \\ & \times\frac{p^4_+}{\sqrt{p^1_+p^3_+}}\frac{(k^3_{\bar z})^2+(k^4_{\bar z})^2+k^4_{\bar z}k^3_{\bar z}}{\langle 12\rangle\langle 23\rangle}\mathrm{tr}\left[B(p^1)B(p^2)B(p^3)B(p^4)\right]\ . \end{aligned} \tag{3.7}$$

The reason this particular combination of $k_{\bar z}$'s appears here is that, given the ordering we chose, after the Mansfield transformation the counterterm ends up being on leg 4, and its line bounds the regions with momenta k_3 and k_3. This is represented pictorially in figure 3.

Although figure 3 might suggest that there is a propagator between the counterterm insertion and the location of the original A, which has now split into three B's, this is of course not the case since the whole expression is a vertex at the same point. We have drawn the diagram in this fashion to emphasise which leg the counterterm is located on after the transformation. On the other hand, this vertex is nonlocal (as discussed above, it was nonlocal even in the original variables, but this is now compounded by the Mansfield coefficients, which contain momenta in the denominator), so this notation serves as a useful reminder of that fact.

It is interesting to note that (3.7) is essentially the same expression as the sum of the two channels with the same region momentum dependence that appear in CQT's

[11] We suppress the overall factor of $-g^2N/(12\pi^2)$ until the end of this section. Also, the integrals are implicitly taken to be on the quantisation surface Σ.

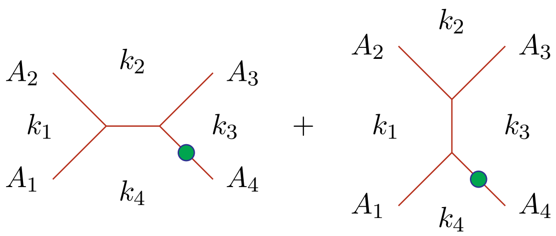


Figure 4: The two diagrams with counterterm insertions on leg 4 that arise in the calculation of CQT, and, combined, add up to the contribution in figure 3.

calculation of this amplitude using tree-level diagrammatics (compare with eq. 83 in [40]), which we illustrate in figure 4. Thus we have a picture where one post-Mansfield transform vertex (with B's) effectively sums *two* tree-level pre-transformation (with A's) Feynman diagrams. This is a first indication that our calculation of the all-plus vertex can be mapped, practically one-to-one, to that of the all-plus amplitude on pp. 22-23 of [40].

Another type of contribution to the vertex arises when we transform *both* of the A's in $\mathcal{L}_{\mathrm{CT}}$. One of the two terms that we find is:

$$
\begin{aligned}
&(\textstyle\int B_2 B_3)(\int B_4 B_1) \\
&= -2g^2 \,\mathrm{tr} \int \mathrm{d}p\, \mathrm{d}p' \delta(p+p') \left[\int \mathrm{d}p^2 \mathrm{d}p^3 \delta(p-p^2-p^3) \frac{p_+}{\sqrt{p_+^2 p_+^3}} \frac{1}{\langle 23 \rangle} B(p^2) B(p^3) \right] \times \\
&\quad \times \left((k_{\bar z}^1)^2 + (k_{\bar z}^3)^2 + k_{\bar z}^1 k_{\bar z}^3 \right) \left[\int \mathrm{d}p^4 \mathrm{d}p^1 \delta(p'-p^4-p^1) \frac{p'_+}{\sqrt{p_+^4 p_+^1}} \frac{1}{\langle 41 \rangle} B(p^4) B(p^1) \right] \\
&= -2g^2 \int \mathrm{d}p^1 \cdots \mathrm{d}p^4 \delta(p^1+p^2+p^3+p^4) \frac{(p_+^2+p_+^3)(p_+^1+p_+^4)}{\sqrt{p_+^1 p_+^2 p_+^3 p_+^4}} \frac{\left((k_{\bar z}^1)^2 + (k_{\bar z}^3)^2 + k_{\bar z}^1 k_{\bar z}^3 \right)}{\langle 23 \rangle \langle 41 \rangle} \\
&\qquad\qquad \times \mathrm{tr} \left[B(p^1) B(p^2) B(p^3) B(p^4) \right] .
\end{aligned} \tag{3.8}
$$

This contribution can also be mapped to one of the two terms with bubbles on internal lines in CQT.

We can now tabulate all the terms that we obtain in this way by making the schematic form (3.6) precise. Since the delta-function and trace over B parts are the same for all these terms, in table 1 we just list the rest of the integrand.

To obtain the final form of the vertex, we are now instructed to sum over all these contributions. Thus we can write

$$
\mathcal{L}_{++++}(B) = 2g^2 \int \mathrm{d}p^1 \mathrm{d}p^2 \mathrm{d}p^3 \mathrm{d}p^4 \delta(p^1+p^2+p^3+p^4)\ \mathcal{V}^{(4)}\ \mathrm{tr}[B(p^1)B(p^2)B(p^3)B(p^4)] \tag{3.9}
$$

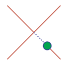
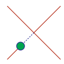
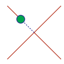
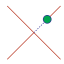
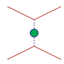
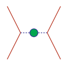
Schematic form	Pictorial form	Integrand
$(\int B_1 B_2 B_3) B_4$		$\frac{p_+^4}{\sqrt{p_+^1 p_+^3}} \frac{k_3^2+k_4^2+k_3 k_4}{\langle 12\rangle\langle 23\rangle}$
$(\int B_2 B_3 B_4) B_1$		$\frac{p_+^1}{\sqrt{p_+^2 p_+^4}} \frac{k_1^2+k_4^2+k_1 k_4}{\langle 23\rangle\langle 34\rangle}$
$(\int B_3 B_4 B_1) B_2$		$\frac{p_+^2}{\sqrt{p_+^3 p_+^1}} \frac{k_1^2+k_2^2+k_2 k_1}{\langle 34\rangle\langle 41\rangle}$
$(\int B_4 B_1 B_2) B_3$		$\frac{p_+^3}{\sqrt{p_+^4 p_+^2}} \frac{k_2^2+k_3^2+k_2 k_3}{\langle 41\rangle\langle 12\rangle}$
$(\int B_2 B_3)(\int B_4 B_1)$		$-\frac{(p_+^2+p_+^3)(p_+^1+p_+^4)}{\sqrt{p_+^1 p_+^2 p_+^3 p_+^4}} \frac{k_1^2+k_3^2+k_1 k_3}{\langle 23\rangle\langle 41\rangle}$
$(\int B_1 B_2)(\int B_3 B_4)$		$-\frac{(p_+^3+p_+^3)(p_+^2+p_+^1)}{\sqrt{p_+^1 p_+^2 p_+^3 p_+^4}} \frac{k_4^2+k_2^2+k_4 k_2}{\langle 34\rangle\langle 12\rangle}$

Table 1: The various contributions to the all-plus four-point vertex. Note that we use the simplifying notation $k_i := k_{\bar{z}}^i$.

where $\mathcal{V}^{(4)}$ is given by the following expression:[12]

$$
\begin{aligned}
\mathcal{V}^{(4)} =& \frac{1}{\sqrt{p_+^1 p_+^2 p_+^3 p_+^4}} \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \times \\
& \times \Big[p_+^4 \sqrt{p_+^2 p_+^4}(k_3^2 + k_4^2 + k_3 k_4)\langle 34\rangle\langle 41\rangle + p_+^1 \sqrt{p_+^1 p_+^3}(k_1^2 + k_4^2 + k_1 k_4)\langle 12\rangle\langle 41\rangle \\
& + p_+^2 \sqrt{p_+^2 p_+^4}(k_2^2 + k_1^2 + k_2 k_1)\langle 12\rangle\langle 23\rangle + p_+^3 \sqrt{p_+^3 p_+^1}(k_3^2 + k_2^2 + k_2 k_3)\langle 23\rangle\langle 34\rangle \\
& - (p_+^2 + p_+^3)(p_+^1 + p_+^4)(k_1^2 + k_3^2 + k_1 k_3)\langle 12\rangle\langle 34\rangle \\
& - (p_+^3 + p_+^4)(p_+^2 + p_+^1)(k_4^2 + k_2^2 + k_4 k_2)\langle 23\rangle\langle 41\rangle \Big] .
\end{aligned}
\tag{3.10}
$$

Comparing this to the expected answer (2.22), we see that the (quadratic) antiholomorphic momentum dependence should arise from the various $k_{\bar{z}}$ factors in (3.10). In [40], CQT start from essentially the same expression and demonstrate that it gives the correct result for the all-plus amplitude. Therefore, following practically the same steps as those authors, we can easily see that we obtain the expected answer. However, since we would like to find the full vertex $\mathcal{V}$, we will need to keep off-shell information, and so we will choose a slightly different route.

[12] For the sake of brevity we omit a subscript $\bar{z}$ in the region momenta appearing in (3.10).

The main complication in bringing (3.10) into a manageable form is clearly the presence of the region momenta. We would like to disentangle their effects as cleanly as possible. Therefore, our derivation will proceed by the following steps:

1. First, we will show that (3.10) can be manipulated so that the quadratic dependence on region momenta drops out, leaving only terms linear in the region momenta.

2. Second, we will decompose the resulting expression into a part that depends on the region momenta and one that does not. The k-dependent part turns out to have a very simple form, and vanishes on-shell.

3. Finally, we will show that the k-independent part reduces to the known amplitude.

For the first step, we will need the following identity, which is proved in appendix B:

$$\begin{aligned} p_+^4\sqrt{p_+^2p_+^4}\langle 34\rangle\langle 41\rangle+p_+^1\sqrt{p_+^1p_+^3}\langle 12\rangle\langle 41\rangle+p_+^2\sqrt{p_+^2p_+^4}\langle 12\rangle\langle 23\rangle+p_+^3\sqrt{p_+^3p_+^1}\langle 23\rangle\langle 34\rangle \\ -(p_+^2+p_+^3)(p_+^1+p_+^4)\langle 12\rangle\langle 34\rangle-(p_+^3+p_+^4)(p_+^2+p_+^1)\langle 23\rangle\langle 41\rangle = 0 \end{aligned} \tag{3.11}$$

Also, using the shorthand notation $K_{ij} := (k_{\bar{z}}^i)^2+(k_{\bar{z}}^j)^2+k_{\bar{z}}^ik_{\bar{z}}^j$: we note the following very useful identity:

$$K_{ij} = K_{ik}+(k_{\bar{z}}^j-k_{\bar{z}}^k)(k_{\bar{z}}^i+k_{\bar{z}}^j+k_{\bar{z}}^k) = K_{ik}+(k_{\bar{z}}^j-k_{\bar{z}}^k)l_{ijk} \tag{3.12}$$

where $1 \leq k \leq n$ and $l_{ijk} = k_{\bar{z}}^i+k_{\bar{z}}^j+k_{\bar{z}}^k$. Noting that, for $j > k$, $k_{\bar{z}}^j-k_{\bar{z}}^k = p_{\bar{z}}^{k+1}+p_{\bar{z}}^{k+2}+\cdots p_{\bar{z}}^j$, we can use this to rewrite all the region momentum combinations appearing in (3.10) in the following way:

$$\begin{aligned} K_{34} &= \frac{1}{4}\left(K_{12}+K_{23}+K_{34}+K_{41}+(\bar{p}_3+\bar{p}_4)(l_{124}+l_{234})+2(\bar{p}_2+\bar{p}_3)l_{134}\right) \\ K_{14} &= \frac{1}{4}\left(K_{12}+K_{23}+K_{34}+K_{41}-(\bar{p}_2+\bar{p}_3)(l_{134}+l_{123})+2(\bar{p}_3+\bar{p}_4)l_{124}\right) \\ K_{12} &= \frac{1}{4}\left(K_{12}+K_{23}+K_{34}+K_{41}-(\bar{p}_3+\bar{p}_4)(l_{124}+l_{234})-2(\bar{p}_2+\bar{p}_3)l_{123}\right) \\ K_{23} &= \frac{1}{4}\left(K_{12}+K_{23}+K_{34}+K_{41}+(\bar{p}_2+\bar{p}_3)(l_{134}+l_{123})-2(\bar{p}_3+\bar{p}_4)l_{234}\right) \\ K_{13} &= \frac{1}{4}\left(K_{12}+K_{23}+K_{34}+K_{41}+(\bar{p}_3-\bar{p}_2)l_{123}+(\bar{p}_1-\bar{p}_4)l_{134}\right) \\ K_{24} &= \frac{1}{4}\left(K_{12}+K_{23}+K_{34}+K_{41}+(\bar{p}_4-\bar{p}_3)l_{234}+(\bar{p}_2-\bar{p}_1)l_{124}\right) \end{aligned} \tag{3.13}$$

where we have introduced the notation $\bar{p}_i = p_{\bar{z}}^i$. We have thus expressed all the quadratic region momentum dependence in terms of the common factor $K_{12}+K_{23}+K_{34}+K_{41}$, and, given (3.11), it is clear that this contribution will vanish.[13]

After this step, we are left with an expression which is linear in the region momenta. We will now proceed in a similar way, and rewrite all the expressions that contain l_{ijk} in

[13] One could have chosen a different combination of the K_{ij}'s, but we find the symmetric choice in (3.13) convenient.

terms of a suitably chosen common factor:

$$
\begin{aligned}
l_{124} + l_{234} =& \frac{3}{2}(k^1_{\bar{z}} + k^2_{\bar{z}} + k^3_{\bar{z}} + k^4_{\bar{z}}) - \frac{1}{2}(p^1_{\bar{z}} + p^3_{\bar{z}}) \\
l_{134} + l_{123} =& \frac{3}{2}(k^1_{\bar{z}} + k^2_{\bar{z}} + k^3_{\bar{z}} + k^4_{\bar{z}}) - \frac{1}{2}(p^2_{\bar{z}} + p^4_{\bar{z}}) \\
2l_{234} =& \frac{3}{2}(k^1_{\bar{z}} + k^2_{\bar{z}} + k^3_{\bar{z}} + k^4_{\bar{z}}) + \frac{1}{2}(2p^2_{\bar{z}} + p^3_{\bar{z}} - p^1_{\bar{z}}) \\
2l_{123} =& \frac{3}{2}(k^1_{\bar{z}} + k^2_{\bar{z}} + k^3_{\bar{z}} + k^4_{\bar{z}}) + \frac{1}{2}(2p^1_{\bar{z}} + p^2_{\bar{z}} - p^4_{\bar{z}}) \\
2l_{134} =& \frac{3}{2}(k^1_{\bar{z}} + k^2_{\bar{z}} + k^3_{\bar{z}} + k^4_{\bar{z}}) + \frac{1}{2}(2p^3_{\bar{z}} + p^4_{\bar{z}} - p^2_{\bar{z}}) \\
2l_{124} =& \frac{3}{2}(k^1_{\bar{z}} + k^2_{\bar{z}} + k^3_{\bar{z}} + k^4_{\bar{z}}) + \frac{1}{2}(2p^4_{\bar{z}} + p^1_{\bar{z}} - p^3_{\bar{z}})
\end{aligned}
\tag{3.14}
$$

In appendix B we show that the total coefficient of the common $(k^1_{\bar{z}} + k^2_{\bar{z}} + k^3_{\bar{z}} + k^4_{\bar{z}})$ factor is

$$
\begin{aligned}
&\frac{3}{8}\Big[p^4_+\sqrt{p^2_+p^4_+}(+(\bar{p}_3+\bar{p}_4)+(\bar{p}_2+\bar{p}_3))\langle 34\rangle\langle 41\rangle+p^1_+\sqrt{p^1_+p^3_+}(-(\bar{p}_2+\bar{p}_3)+(\bar{p}_3+\bar{p}_4))\langle 12\rangle\langle 41\rangle \\
&+p^2_+\sqrt{p^2_+p^4_+}(-(\bar{p}_3+\bar{p}_4)-(\bar{p}_2+\bar{p}_3))\langle 12\rangle\langle 23\rangle+p^3_+\sqrt{p^3_+p^1_+}(+(\bar{p}_2+\bar{p}_3)-(\bar{p}_3+\bar{p}_4))\langle 23\rangle\langle 34\rangle \\
&-(p^2_+ + p^3_+)(p^1_+ + p^4_+)\left(\frac{1}{2}(\bar{p}_3 - \bar{p}_2) + \frac{1}{2}(\bar{p}_1 - \bar{p}_4)\right)\langle 12\rangle\langle 34\rangle \\
&-(p^3_+ + p^4_+)(p^2_+ + p^1_+)\left(\frac{1}{2}(\bar{p}_4 - \bar{p}_3) + \frac{1}{2}(\bar{p}_2 - \bar{p}_1)\right)\langle 23\rangle\langle 41\rangle\Big] = \\
&= -\frac{3}{16}[(12) + (23) + (34) + (41)]\sum_{i=i}^{4}\frac{(p_i)^2}{p^i_+},
\end{aligned}
\tag{3.15}
$$

where $(p_i)^2$ is the full covariant momentum squared, and $(ij) = p^i_+ p^j_z - p^j_+ p^i_z$. Thus we see that the complete dependence on the region momenta can be rewritten as follows:

$$
\mathcal{V}^{(4)}_k = -\frac{3}{16}\frac{(12) + (23) + (34) + (41)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left[\sum_{i=1}^{4} k^i_{\bar{z}}\right]\sum_{i=i}^{4}\frac{(p_i)^2}{p^i_+}. \tag{3.16}
$$

It is rather satisfying that the region momentum dependence of the vertex takes this simple form, which clearly vanishes when the external legs are on-shell, and thus will not contribute to the all-plus amplitudes.

Having completely disentangled the region momenta $k_{\bar{z}}$ from the actual momenta $p_{\bar{z}}$, we will now focus on the terms containing only the latter, which were produced during the decompositions in (3.14). After a few simple manipulations, they can be rewritten as[14]

$$\begin{aligned}
V_p^{(4)} = \frac{1}{8}\Bigg[& p_+^4\sqrt{p_+^2 p_+^4}\,[(\bar p_1+\bar p_2)(\bar p_1-\bar p_2)+(\bar p_3+\bar p_2)(\bar p_3-\bar p_2)]\langle 34\rangle\langle 41\rangle \\
& + p_+^1\sqrt{p_+^1 p_+^3}\,[(\bar p_2+\bar p_3)(\bar p_2-\bar p_3)+(\bar p_4+\bar p_3)(\bar p_4-\bar p_3)]\langle 41\rangle\langle 12\rangle \\
& + p_+^2\sqrt{p_+^2 p_+^4}\,[(\bar p_3+\bar p_4)(\bar p_3-\bar p_4)+(\bar p_1+\bar p_4)(\bar p_1-\bar p_4)]\langle 12\rangle\langle 23\rangle \\
& + p_+^3\sqrt{p_+^3 p_+^1}\,[(\bar p_4+\bar p_1)(\bar p_4-\bar p_1)+(\bar p_2+\bar p_1)(\bar p_2-\bar p_1)]\langle 23\rangle\langle 34\rangle \\
& - (p_+^2+p_+^3)(p_+^1+p_+^4)[(\bar p_3-\bar p_2)(\bar p_1-\bar p_4)-(\bar p_1+\bar p_2)^2]\langle 12\rangle\langle 34\rangle \\
& - (p_+^3+p_+^4)(p_+^2+p_+^1)[(\bar p_4-\bar p_3)(\bar p_2-\bar p_1)-(\bar p_2+\bar p_3)^2]\langle 23\rangle\langle 41\rangle \Bigg] \;.
\end{aligned} \tag{3.17}$$

This expression, together with (3.16) is our proposal for the off-shell four-point all-plus vertex that should be part of the MHV-rules formalism at the quantum level. It would be very interesting to elucidate its structure and bring it into a more compact form. For the moment, however, we will be content to demonstrate that (3.17) is equal on shell to the sought-for amplitude.

To that end, we will follow a similar approach to CQT, and rewrite all the holomorphic spinor brackets in terms of the following three: $\langle 12\rangle\langle 34\rangle, \langle 23\rangle\langle 41\rangle, \langle 12\rangle\langle 41\rangle$. To achieve this, we use momentum conservation and a certain cyclic identity (see appendix A) to write

$$\begin{aligned}
p_+^4\sqrt{p_+^2p_+^4}\langle 34\rangle\langle 41\rangle &= p_+^4\sqrt{p_+^4}\left(-\sqrt{p_+^3}\langle 42\rangle-\sqrt{p_+^4}\langle 23\rangle\right)\langle 41\rangle \\
&= \left[-p_+^4\sqrt{p_+^3p_+^4}\langle 42\rangle-(p_+^4)^2\right]\langle 41\rangle \\
&= \left[-p_+^4\sqrt{p_+^3}\left(-\sqrt{p_+^1}\langle 12\rangle-\sqrt{p_+^3}\langle 32\rangle\right)-(p_+^4)^2\langle 23\rangle\right]\langle 41\rangle \\
&= p_+^4\sqrt{p_+^3p_+^1}\langle 12\rangle\langle 41\rangle - p_+^4(p_+^4+p_+^3)\langle 23\rangle\langle 41\rangle \;.
\end{aligned} \tag{3.18}$$

In a similar way, we can show that

$$\begin{aligned}
p_+^2\sqrt{p_+^2p_+^4}\langle 12\rangle\langle 23\rangle &= p_+^2\sqrt{p_+^3p_+^1}\langle 12\rangle\langle 41\rangle - p_+^2(p_+^2+p_+^3)\langle 34\rangle\langle 12\rangle \;, \\
p_+^3\sqrt{p_+^1p_+^3}\langle 23\rangle\langle 34\rangle &= -\left[p_+^3(p_+^3+p_+^2)\langle 12\rangle\langle 34\rangle - p_+^3(p_+^1+p_+^2)\langle 23\rangle\langle 41\rangle + p_+^3\sqrt{p_+^1p_+^3}\langle 12\rangle\langle 14\rangle\right] \;.
\end{aligned} \tag{3.19}$$

Collecting all the terms together, and manipulating the resulting expressions, it is straightforward to show that (3.17) simplifies to just

$$\begin{aligned}
V_p^{(4)} = \frac{1}{4}\Bigg[& \langle 23\rangle\langle 41\rangle\{34\}(p_+^1+p_+^2)[(\bar p_1-\bar p_2)-(\bar p_2+\bar p_3)] \\
& + \langle 12\rangle\langle 34\rangle\{23\}(p_+^2+p_+^3)[(\bar p_1+\bar p_2)+(\bar p_1-\bar p_4)] \\
& + \langle 12\rangle\langle 41\rangle\sqrt{p_+^3p_+^1}\,[(\bar p_1+\bar p_2)(\{41\}+\{32\})+(\bar p_2+\bar p_3)(\{12\}+\{43\})]\Bigg] \;,
\end{aligned} \tag{3.20}$$

[14] We write $V^{(4)} = \sqrt{p_+^1p_+^2p_+^3p_+^4}\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle\mathcal{V}^{(4)}$.

where we use the notation [34] $\{ij\} = p^i_+ p^j_{\bar z} - p^j_+ p^i_{\bar z} = (1/\sqrt{2})\sqrt{p^i_+ p^j_+}[ij]$. Converting to the usual antiholomorphic bracket notation, we rewrite (3.20) as

$$\begin{aligned} V_p^{(4)} = \frac{1}{4\sqrt{2}}\Big[&\langle 23\rangle\langle 41\rangle\sqrt{p^3_+p^4_+}[34](p^1_+ + p^2_+)[(\bar p_1 - \bar p_2) - (\bar p_2 + \bar p_3)] \\ &+ \langle 12\rangle\langle 34\rangle\sqrt{p^2_+p^3_+}[23](p^2_+ + p^3_+)[(\bar p_1 + \bar p_2) + (\bar p_1 - \bar p_4)] \\ &+ \langle 12\rangle\langle 41\rangle\Big[(\bar p_1 + \bar p_2)(p^1_+\sqrt{p^3_+p^4_+}[41] + p^2_+\sqrt{p^2_+p^1_+}[32]) \\ &+ (\bar p_2 + \bar p_3)\Big(p^1_+\sqrt{p^2_+p^3_+}[12] + p^3_+\sqrt{p^1_+p^4_+}[43]\Big)\Big]\Big] . \end{aligned} \tag{3.21}$$

Note that so far this expression is completely off shell. We will now show that on shell it reduces to the known result (2.22). In doing this we will keep track of the p^2 terms that appear when applying momentum conservation in the form

$$\sum_k \langle ik\rangle[kj] = \sqrt{p^i_+ p^j_+}\sum_k \frac{(p_k)^2}{p^k_+} . \tag{3.22}$$

These terms are collected in appendix B.

We start by rewriting each of the terms in the last two lines of (3.21) as follows

$$\begin{aligned} \langle 12\rangle\langle 41\rangle[41]\ p^1_+\sqrt{p^3_+p^4_+}(\bar p_1 + \bar p_2) &= -\langle 23\rangle\langle 41\rangle[34]\ p^1_+\sqrt{p^3_+p^4_+}(\bar p_1 + \bar p_2) \\ \langle 12\rangle\langle 41\rangle[32]\ p^3_+\sqrt{p^1_+p^2_+}(\bar p_1 + \bar p_2) &= -\langle 12\rangle[32]\langle 42\rangle p^2_+p^3_+(\bar p_1 + \bar p_2) \\ &\quad - \langle 12\rangle\langle 34\rangle[23]\ p^3_+\sqrt{p^2_+p^3_+}(\bar p_1 + \bar p_2) \\ \langle 12\rangle\langle 41\rangle[12]\ p^1_+\sqrt{p^2_+p^3_+}(\bar p_2 + \bar p_3) &= -\langle 12\rangle\langle 34\rangle[23]\ p^1_+\sqrt{p^2_+p^3_+}(\bar p_2 + \bar p_3) \\ \langle 12\rangle\langle 41\rangle[43]\ p^3_+\sqrt{p^1_+p^4_+}(\bar p_2 + \bar p_3) &= -\langle 41\rangle\langle 23\rangle[34]\ p^3_+\sqrt{p^3_+p^4_+}(\bar p_2 + \bar p_3) \\ &\quad - \langle 41\rangle[43]\langle 42\rangle p^4_+p^3_+(\bar p_2 + \bar p_3) . \end{aligned} \tag{3.23}$$

We also transform the $\langle 12\rangle\langle 34\rangle$ term using the Schouten identity and also momentum conservation,

$$\langle 12\rangle\langle 34\rangle[23]\sqrt{p^2_+p^3_+} = \langle 23\rangle\langle 41\rangle[34]\sqrt{p^3_+p^4_+} + \langle 14\rangle\langle 23\rangle[13]\sqrt{p^1_+p^3_+} - \langle 13\rangle\langle 42\rangle[23]\sqrt{p^2_+p^3_+} , \tag{3.24}$$

and add up all contributions to the $\langle 23\rangle\langle 41\rangle$ term, which are

$$\begin{aligned} &\frac{1}{4\sqrt{2}}\langle 23\rangle\langle 41\rangle[34]\sqrt{p^3_+p^4_+}\left[4(p^2_+\bar p_1 - p^1_+\bar p_2) + 2(p^3_+\bar p_1 - p^1_+\bar p_3)\right] \\ &\qquad = \frac{1}{4\sqrt{2}}\langle 23\rangle\langle 41\rangle[34]\sqrt{p^3_+p^4_+}[4\{21\} + 2\{31\}] . \end{aligned} \tag{3.25}$$

Converting to the spinor bracket, the first of these terms is

$$-\frac{1}{2}\sqrt{p^1_+p^2_+p^3_+p^4_+}[12]\langle 23\rangle[34]\langle 41\rangle , \tag{3.26}$$

while the remaining terms from (3.23) and (3.24) combine to give

$$\begin{aligned}\Big(\langle 14\rangle\langle 23\rangle[13]\sqrt{p_+^1 p_+^3}-\langle 13\rangle\langle 42\rangle[23]\sqrt{p_+^2 p_+^3}\Big)&(p_+^2+p_+^3)[(\bar p_1+\bar p_2)+(\bar p_1-\bar p_4)]\\ +\langle 12\rangle[32]\langle 42\rangle p_+^2[p_+^2(\bar p_1+\bar p_2)-p_+^4(\bar p_2+\bar p_3)] &= -\langle 14\rangle[13]\langle 12\rangle p_+^3(p_+^2+p_+^3)[(\bar p_1+\bar p_2)+(\bar p_1-\bar p_4)]\\ &\quad +\langle 12\rangle[32]\langle 42\rangle p_+^2[p_+^2(\bar p_1+\bar p_2)-p_+^4(\bar p_2+\bar p_3)]\\ &= -\langle 14\rangle[13]\langle 12\rangle p_+^3(2(p_+^2+p_+^3)\bar p_1-2p_+^1(\bar p_2+\bar p_3))\\ &= 2\langle 14\rangle[13]\langle 12\rangle p_+^3\{41\} \end{aligned} \tag{3.27}$$

(where we suppress an overall $1/(4\sqrt{2})$) and we see that (3.27) cancels the second term in (3.25), thus showing that (3.26) is the complete on-shell answer. Reintroducing all the prefactors, we thus find that the amplitude is

$$\begin{aligned}\mathcal{A}^{(4)} &= -\frac{g^2N}{12\pi^2}\frac{2g^2}{\sqrt{p_+^1p_+^2p_+^3p_+^4}}\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\times\left[-\frac{1}{2}\sqrt{p_+^1p_+^2p_+^3p_+^4}[12]\langle 23\rangle[34]\langle 41\rangle\right]\\ &= \frac{g^4N}{12\pi^2}\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\ .\end{aligned} \tag{3.28}$$

Now note that, as discussed in appendix A, in order to convert to the usual Yang-Mills theory normalisation we need to send $g \to g/\sqrt{2}$. We conclude that $\mathcal{A}^{(4)}$ gives precisely the result (2.22) for the all-plus scattering amplitude.

3.3 The general all-plus amplitude

We have just given an explicit derivation of the four point all-plus amplitude, from the two-point counterterm (3.3). We will argue in the following that this two-point counterterm contains *all* the all-plus amplitudes.

First, we can see immediately that the counterterm (3.3) has the right kind of structure. Consider the n-point all-plus amplitude [56]:

$$\mathcal{A}^{(n)} = \sum_{1\le i<j<k<l\le n}\frac{\langle ij\rangle[jk]\langle kl\rangle[li]}{\langle 12\rangle\cdots\langle n1\rangle}\ . \tag{3.29}$$

In terms of spinor brackets this amplitude has terms of the form $\langle\ \ \rangle^{2-n}[\ \ ]^2$. A quick look at the Ettle-Morris coefficients shows that, for an n-point vertex coming from $\mathcal{L}_{\rm CT}$, they contribute exactly $2-n$ powers of the spinor brackets $\langle\ \ \rangle$. Furthermore, there are exactly two powers of $[\ \ ]$ coming from the counterterm Lagrangian $\mathcal{L}_{\rm CT}\sim(k_{\bar z}^2)A^2$ — one for each power of k. Thus the general structure of $\mathcal{L}_{\rm CT}$ is appropriate to reproduce (3.29).

Pictorially, we can represent the general n-point amplitude, arising from the counterterm in the new variables, as in figure 5.

JHEP07(2007)002

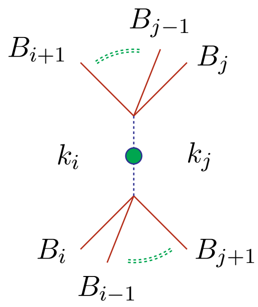


Figure 5: The structure of a generic term contributing to the n-point vertex. All momenta are taken to be outgoing, and all indices are modulo n.

Thus we can write this n-point all-plus vertex as follows:

$$\begin{aligned}\mathcal{A}^{(n)}_{+\cdots+} &= \int_{1\cdots n}\delta(p+p')\sum_{1\le i<j\le n}\Upsilon(p;j+1,\ldots,i)\left((k^i_{\bar z})^2+(k^j_{\bar z})^2+k^i_{\bar z}k^j_{\bar z}\right)\Upsilon(p';i+1,\ldots,j)\times\\ &\quad\times\mathrm{tr}[B_iB_{i+1}\cdots B_jB_{j+1}\cdots B_{i-1}] \qquad (3.30)\\ &= (\sqrt{2}i)^{n-2}\int_{1\cdots n}\delta(p^1+\cdots+p^n)\sum_{1\le i<j\le n}\frac{(p^{j+1}_++\cdots+p^i_+)}{\sqrt{p^{j+1}_+p^i_+}}\frac{1}{\langle j+1,j+2\rangle\cdots\langle i-1,i\rangle}\times\\ &\quad\times\left((k^i_{\bar z})^2+(k^j_{\bar z})^2+k^i_{\bar z}k^j_{\bar z}\right)\frac{(p^{i+1}_++\cdots+p^j_+)}{\sqrt{p^{i+1}_+p^j_+}}\frac{1}{\langle i+1,i+2\rangle\cdots\langle j-1,j\rangle}\mathrm{tr}[B_1\cdots B_n]\,.\end{aligned}$$

Focusing only on the relevant part of the above expression, and ignoring all coefficients, the general structure we obtain is the following:

$$\mathcal{V}^{(n)}_{+\cdots+} = \frac{1}{\langle 12\rangle\cdots\langle n1\rangle}\times\left[\sum_{1\le i<j\le n}\frac{\langle j,j+1\rangle\langle i,i+1\rangle}{\sqrt{p^i_+p^{i+1}_+p^j_+p^{j+1}_+}}(k^j_+-k^i_+)^2((k^i_{\bar z})^2+(k^j_{\bar z})^2+k^i_{\bar z}k^j_{\bar z})\right] \tag{3.31}$$

where we have extracted the denominator at the expense of introducing the two missing holomorphic factors $\langle j,j+1\rangle$ and $\langle i,i+1\rangle$ in the numerator. We also made use of the fact that

$$k^j-k^i = p^{i+1}+p^{i+2}+\cdots+p^j = -(p^{j+1}+p^{j+2}+\cdots+p^i)\,, \tag{3.32}$$

applied to the $+$ components, to rewrite the two p_+ sums in the numerator in terms of the k's (this gives rise to a minus which we suppress).

It is easy to verify that, for $n=4$, this sum reproduces the 6 contributions that appeared in the four-point case, and (as we explicitly showed above) combined to give the expected answer. Therefore, we would like to propose that the vertex (3.31) will reduce on-shell to an expression proportional to (3.29). We will not attempt to prove this statement here,[15] but will instead move on to study the general properties of the n-point expression (3.30).

[15] It is perhaps interesting to remark that the proof would involve converting the double sum in (3.31) to the quadruple sum in (3.29) —a state of affairs which has appeared before in a rather different context [20].

Whilst the explicit calculation for the four point case was rather involved as we saw earlier, the study of the general properties of the n-point amplitudes proves much simpler. In particular, we will show that the collinear and soft limits of the expressions proposed for the n-point case can be very easily shown to be correct. Let us start by introducing some simplifying notation. One can write the change of variables for the A field as

$$A_1 = \mathrm{Y}_{12} B_2 + \mathrm{Y}_{123} B_2 B_3 + \mathrm{Y}_{1234} B_2 B_3 B_4 + \cdots , \tag{3.33}$$

where

$$\mathrm{Y}_{12} = \delta_{12}, \qquad \mathrm{Y}_{123} = \frac{1_+}{(23)}, \qquad \mathrm{Y}_{1234} = \frac{1_+ 3_+}{(23)(34)}, \tag{3.34}$$

and generally

$$\mathrm{Y}_{12\ldots n} = \frac{1_+ 3_+ 4_+ \ldots (n-1)_+}{(23)(34)\ldots(n-1\ n)} \tag{3.35}$$

(for simplicity, we are dropping inconsequential constant factors in this discussion). This notation is similar to that of [34]. Integrations and the insertion of suitable delta functions are understood, and can be illustrated by comparing the short-hand expressions above with the full equations given earlier. It will prove convenient to define

$$K_{ij} = k_i^2 + k_j^2 + k_i k_j, \qquad k_i := k_{\bar{z}}^i. \tag{3.36}$$

We will use the expression $\mathrm{Y}_{\bullet 12\ldots n}$ in the following, where the dot in the first placemark in the Y means that one substitutes in that place the negative of the sum of the other momenta. Then the result which we have proved above for the four point amplitude V_{1234} can be expressed as

$$\begin{aligned} V_{1234} =& K_{43} \mathrm{Y}_{\bullet 4} \mathrm{Y}_{\bullet 123} + K_{14} \mathrm{Y}_{\bullet 1} \mathrm{Y}_{\bullet 234} + K_{21} \mathrm{Y}_{\bullet 2} \mathrm{Y}_{\bullet 341} + K_{32} \mathrm{Y}_{\bullet 3} \mathrm{Y}_{\bullet 412} \\ &+ K_{31} \mathrm{Y}_{\bullet 23} \mathrm{Y}_{\bullet 41} + K_{24} \mathrm{Y}_{\bullet 12} \mathrm{Y}_{\bullet 34} , \end{aligned} \tag{3.37}$$

or very simply

$$V_{1234} = \sum_{1 \le i < j \le 4} K_{ij} \mathrm{Y}_{\bullet\, j+1\ldots i} \mathrm{Y}_{\bullet\, i+1\ldots j} \ . \tag{3.38}$$

It is clear that the general conjecture that all the n-point all plus amplitudes are generated from the two-point counterterm (3.3) translates into the proposal that the n-point all-plus amplitude $V_{12\ldots n}$ is given by

$$V_{12\ldots n} = \sum_{1 \le i < j \le n} K_{ij} \mathrm{Y}_{\bullet\, j+1\ldots i} \mathrm{Y}_{\bullet\, i+1\ldots j} , \tag{3.39}$$

Let us now show that the expression on the right-hand side of (3.39) has precisely the same soft and collinear limits as the known amplitude on the left-hand side.

Collinear limits. Under the collinear limit

$$p_i \to zP , \qquad p_{i+1} \to (1-z)P , \qquad P^2 \to 0 , \tag{3.40}$$

the n-point amplitude $V_{12\dots n}$ behaves as

$$V_{12\dots n} \to \frac{1}{z(1-z)} \frac{i_+}{(i\,i+1)}\; V_{12\dots i\;i+2\dots n}\,, \tag{3.41}$$

where we relabel $P \to p_i$ after the limit is taken (the i_+ and $(i\;i+1)$ factors involve momenta rather than spinors, which is why the z-dependent factor is $1/z(1-z)$, rather than the conventional $1/\sqrt{z(1-z)}$).

Consider the behaviour of the right-hand side of (3.39) under the limit (3.40). The first point is that if the indices $i, i+1$ lie on different Y's, then there are no poles generated in this collinear limit. This is clear from the explicit expressions for the Y's in (3.35). Thus we may ignore any terms of this type. It is then immediate from the explicit forms of the Y's that

$$\mathrm{Y}_{12\dots s} \to \frac{1}{z(1-z)} \frac{i_+}{(i\,i+1)}\; \mathrm{Y}_{12\dots i\;i+2\dots s}\,, \tag{3.42}$$

for any $i = 2, \dots s-1$, with $s \le n$ (the first index in Y never contributes in a collinear limit, as one can see from the conjecture (3.39)). Thus we see that the Y expressions have the right sort of collinear behaviour. It is straightforward to see that the K coefficients in (3.39) also get relabelled correctly in the collinear limit; they are not explicitly involved as they refer to pairs of momenta attached to different Y fields, and as we saw, these do not contribute.

It is then immediate to see that the summation over the products of Y's in (3.39) reduces correctly in the collinear limit to the required summation over products of Y's with one fewer leg in total. Hence the proposal (3.39) for the amplitude has precisely the same collinear limits as the physical amplitude.

Soft limits. We also find that there is a simple derivation of the soft limits of the expression in (3.39). In the soft limit

$$p_j \to 0\,, \tag{3.43}$$

the n-point amplitude $V_{12\dots n}$ behaves as

$$V_{12\dots n} \to S(j)\; V_{12\dots j-1\;j+1\dots n}\,, \tag{3.44}$$

where we assume cyclic ordering as usual, so that, for example, $p_{n+1} = p_1$. The soft function $S(j)$ is given in terms of the momentum brackets by

$$S(j) = \frac{j_+(j-1\,j+1)}{(j-1\,j)\,(j\,j+1)}\,. \tag{3.45}$$

The Y functions have a simple behaviour under soft limits. One has immediately that in the soft limit $p_j \to 0$,

$$\mathrm{Y}_{12\dots s} \to S(j)\; \mathrm{Y}_{12\dots j-1\;j+1\dots s}\,, \tag{3.46}$$

for $j = 3, \dots s-1$ (with $s \le n$). For the soft limits corresponding to the case missing in the above, we need the results

$$\mathrm{Y}_{\bullet s+1\dots j} = \mathrm{Y}_{\bullet s+1\dots j-1} \frac{(j-1)_+}{(j-1\,j)}, \qquad \mathrm{Y}_{\bullet j\dots s} = \mathrm{Y}_{\bullet j+1\dots s} \frac{(j+1)_+}{(j\,j+1)}\,, \tag{3.47}$$

which follow from the definitions of the Y's, and

$$\frac{(j+1)_+}{(j\,j+1)}+\frac{(j-1)_+}{(j-1\,j)}=\frac{j_+(j-1\,j+1)}{(j-1\,j)\;(j\,j+1)}=S(j)\,, \tag{3.48}$$

which follows from the cyclic identity $i_+(jk)+j_+(ki)+k_+(ij)=0$. Finally, from relabelling the K's we have in the soft limit that $K_{sj}\to K_{sj-1}$. Then it follows that in the soft limit

$$K_{sj}\;\mathrm{Y}_{\bullet s+1\ldots j}\;\mathrm{Y}_{\bullet j+1\ldots s}+K_{sj-1}\;\mathrm{Y}_{\bullet s+1\ldots j-1}\;\mathrm{Y}_{\bullet j\ldots s}\to S(j)K_{sj-1}\;\mathrm{Y}_{\bullet s+1\ldots j-1}\;\mathrm{Y}_{\bullet j+1\ldots s}\,, \tag{3.49}$$

as required.

Again, it is then easy to see that the summation over the products of Y's in (3.39) reduces correctly in the soft limit to the required summation over products of Y's with one fewer leg in total. Hence the proposal (3.39) for the amplitude has precisely the same soft limits as the physical amplitude.

4. Discussion

Whilst new, twistor-inspired methods for calculating amplitudes in gauge theory have led to much progress, the lack of a systematic action-based formulation which incorporates these new ideas has been an impediment to further developments. MHV diagrams have the two advantages of being closely allied to the twistor picture, as well as providing an explicit realisation of the dispersion and phase space integrals fundamental to unitarity-based methods. However, without an action formalism, standard MHV methods have so far been mainly restricted to massless theories at one-loop level, and to the cut-constructible parts of amplitudes.

The advent of a classical MHV Lagrangian for gauge theory, derived from lightcone YM theory [32 – 34], provides the basis for transcending these limitations. In order for this to be realised, it is necessary to describe the quantum MHV theory. What we have done in this paper is to investigate this quantum theory. Using the regularisation methods of [39 – 41], we have provided arguments that the simplest one-loop counterterm in the quantum MHV theory — a two point vertex — provides an extraordinarily concise generating function for the infinite sequence of one-loop, all-plus helicity amplitudes in YM theory. We showed this by explicit calculation for the four-point case, and then proved that the soft and collinear limits of the conjectured n-point amplitude precisely matched those of the correct answer.

We would like to emphasise that the simplicity of our approach — which reduced the calculations of the loop amplitudes we considered to tree-level algebraic manipulations — is largely due to the four-dimensional nature of the regularisation scheme we employed. By staying in four dimensions, we preserve the appealing features of the inherently four-dimensional field redefinition of [32, 33].

Based upon this result, it is very natural to conjecture that the full quantum YM theory is correctly described by this quantum MHV Lagrangian. The correct ingredients appear to be present. For example, in the approach of [39 – 41] there arise one-loop counterterms with helicities $(++), (++-), (--), (--+)$. We studied the $(++)$ counterterm in this paper, arguing that when expressed in the $(B,\bar{B})$ variables this generates the full set

JHEP07(2007)002

of all-plus amplitudes. Transforming the $(++-)$ counterterm to $(B, \bar{B})$ variables will generate an infinite sequence of single-minus vertices. There will be other contributions to single-minus vertices from combinations of all-plus vertices and MHV vertices. It would be surprising if the combined contributions of these did not lead to the correct YM single-minus expressions. Certainly all of these have the correct powers of spinor brackets for this to be the case.

Transforming the $(--)$ and $(--+)$ counterterms to $(B, \bar{B})$ variables will lead to new contributions to MHV vertices.[16] The MHV vertices from the classical MHV Lagrangian only generate the cut-constructible parts of YM loop amplitudes, such as the one-loop MHV amplitude. These new contributions might be expected to lead to the missing, rational parts. This would also potentially explain why in [57] the combination of all-plus vertices with MHV tree vertices did not yield the correct single-minus amplitudes — these additional MHV contributions are missing.

Further evidence for the conjecture that the quantum MHV Lagrangian is equivalent to quantum YM theory would be welcome. One could start with seeking explicit proofs of the above proposals. One can also investigate beyond massless one-loop gauge theory — an advantage of the Lagrangian approach is that the inclusion of masses, and of fermions and scalars, is in principle clear. There are other issues raised by this work. It is plausible that the potential quantum versions of the twistor space formulations of gauge theory [58–60] are most likely to be allied to the quantum theory discussed here — one simple reason for believing this is that the regularisation employed here keeps one in four dimensions. Perhaps there are simple twistor space analogues of the counterterms discussed above.

Finally, although for our purposes the lightcone worldsheet approach to perturbative gauge theory provided simply the motivation for a particular choice of regularisation scheme, we believe that it would be fruitful to further explore possible connections between that framework and the twistor string programme.

Note added. We would like to thank Paul Mansfield and Tim Morris for having informed us that they have recently been pursuing research related to that presented in this paper. Their work, which is complementary to ours in that it employs dimensional regularisation, has now appeared in [61].

Acknowledgments

It is a pleasure to thank Paul Heslop, Gregory Korchemsky, Paul Mansfield, Tim Morris and Adele Nasti for discussions. We would like to thank PPARC for support under the Rolling Grant PP/D507323/1 and the Special Programme Grant PP/C50426X/1. The work of GT is supported by an EPSRC Advanced Fellowship EP/C544242/1 and by an EPSRC Standard Research Grant EP/C544250/1.

[16] In the MHV case there are additional counterterms noted in [41] which may also need to be taken into account in future discussions.

A. Notation

Lightcone conventions. Here we summarise our lightcone conventions. We start off by introducing lightcone coordinates

$$x^{\pm} := \frac{x^0 \pm x^3}{\sqrt{2}}\,, \quad x^z := \frac{x^1 + ix^2}{\sqrt{2}}\,, \quad x^{\bar{z}} := \frac{x^1 - ix^2}{\sqrt{2}}\,. \tag{A.1}$$

We also have $x^+ = x_-$, $x^z = -x_{\bar{z}}$, and so on. The scalar product between two vectors A and B is written as

$$A \cdot B := A_+B_- + A_-B_+ - A_zB_{\bar{z}} - A_{\bar{z}}B_z\,. \tag{A.2}$$

We choose x^- as our lightcone time coordinate, therefore the lightcone gauge used in this paper is defined by

$$A^- = 0\,. \tag{A.3}$$

This condition can be written as $\eta \cdot A = 0$, where η is a constant null vector, chosen to have components $\eta := (1/\sqrt{2}, 0, 0, 1/\sqrt{2})$ (hence $\eta_- = 1$, $\eta_+ = \eta_z = \eta_{\bar{z}} = 0$).

To any four-vector p we associate the bispinor $p_{a\dot{a}}$ defined by

$$p_{a\dot{a}} \;:=\; \sqrt{2}\begin{pmatrix} p_- & -p_z \\ -p_{\bar{z}} & p_+ \end{pmatrix}\,. \tag{A.4}$$

We also define holomorphic and anti-holomorphic spinors as

$$\lambda_a \;:=\; \frac{2^{\frac{1}{4}}}{\sqrt{p_+}}\begin{pmatrix} -p_z \\ p_+ \end{pmatrix}\,, \qquad \tilde{\lambda}_{\dot{a}} \;:=\; \frac{2^{\frac{1}{4}}}{\sqrt{p_+}}\begin{pmatrix} -p_{\bar{z}} \\ p_+ \end{pmatrix}\,, \tag{A.5}$$

from which it follows that

$$\lambda_a\tilde{\lambda}_{\dot{a}} \;:=\; \sqrt{2}\begin{pmatrix} \frac{p_z p_{\bar{z}}}{p_+} & -p_z \\ -p_{\bar{z}} & p_+ \end{pmatrix}\,. \tag{A.6}$$

This is of course consistent with the on-shell condition $p_- = p_z p_{\bar{z}}/p_+$. Furthermore, comparing (A.4) and (A.6) and choosing η as specified earlier, we see that a generic off-shell vector p can be decomposed as

$$p \;=\; \lambda\tilde{\lambda} + z\eta\,, \tag{A.7}$$

where

$$z \;=\; \frac{p_-p_+ - p_zp_{\bar{z}}}{p_+\eta_-} \;=\; \frac{p^2}{2(p\cdot\eta)}\,. \tag{A.8}$$

(A.7) and (A.8) are the familiar decompositions of off-shell vectors in the MHV literature [62, 17, 63, 15].

The off-shell holomorphic spinor product is defined as:

$$\langle ij\rangle = \sqrt{2}\,\frac{p_+^i p_z^j - p_+^j p_z^i}{\sqrt{p_+^i p_+^j}}\,, \tag{A.9}$$

whereas for the antiholomorphic spinors we define

$$[ij] = \sqrt{2}\,\frac{p_+^i p_{\bar z}^j - p_+^j p_{\bar z}^i}{\sqrt{p_+^i p_+^j}}\ . \tag{A.10}$$

In these conventions, one finds

$$2(p^i\cdot p^j) \ = \ \langle i\,j\rangle\,[i\,j] + \left(\frac{p_+^j}{p_+^i}\right)(p^i)^2 + \left(\frac{p_+^i}{p_+^j}\right)(p^j)^2\,, \tag{A.11}$$

or, in the case where p^i and p^j are on shell, $2(p^i\cdot p^j) = \langle i\,j\rangle\,[i\,j]$. In the standard QCD literature conventions it is customary to define $2(p^i\cdot p^j) = \langle i\,j\rangle\,[j\,i]$; this can be obtained by simply re-defining the inner product of two anti-holomorphic spinors, $[i\,j]$, to be the negative of the right hand side of (A.10).

Useful identities. The form (A.9) is very convenient for deriving identities for $\langle ij\rangle$ that also involve the p_+ components. For instance, one has:

$$\begin{aligned}&\sqrt{p_+^i}\langle jk\rangle + \sqrt{p_+^j}\langle ki\rangle + \sqrt{p_+^k}\langle ij\rangle \\ &= \sqrt{2}\,\frac{p_+^i(p_+^j p_z^k - p_+^k p_z^j)}{\sqrt{p_+^i p_+^j p_+^k}} + \sqrt{2}\,\frac{p_+^j(p_+^k p_z^i - p_+^i p_z^k)}{\sqrt{p_+^i p_+^j p_+^k}} + \sqrt{2}\,\frac{p_+^k(p_+^i p_z^j - p_+^j p_z^i)}{\sqrt{p_+^i p_+^j p_+^k}} = 0\ .\end{aligned} \tag{A.12}$$

It is also easy to see how to apply momentum conservation, take say $\langle ij\rangle$, and substitute

$$p^j = -\sum_{k\neq j} p^k \qquad \text{(for each component)}. \tag{A.13}$$

Then we have

$$\sqrt{p_+^j}\langle ij\rangle = \sqrt{2}\frac{p_+^i(-\sum_{k\neq j}p_z^k)+(\sum_{k\neq j}p_+^k)p_z^i}{\sqrt{p_+^i}} = -\sqrt{2}\sum_{k\neq j}\sqrt{p_+^k}\,\frac{p_+^i p_z^k - p_+^k p_z^i}{\sqrt{p_+^i p_+^k}} = \sum_{k\neq j}\sqrt{p_+^k}\langle ki\rangle\ . \tag{A.14}$$

We have also used the momentum bracket notation from [34]

$$(ij) = p_+^i p_z^j - p_+^j p_z^i\,, \quad \{ij\} = p_+^i p_{\bar z}^j - p_+^j p_{\bar z}^i\ . \tag{A.15}$$

Lightcone Yang-Mills action. Here we give the form of the lightcone Yang-Mills action that we use in this paper. As discussed in more detail in [35], starting from the YM Lagrangian $-(1/4)\,\mathrm{tr}F^2$, imposing the lightcone gauge (A.3), and integrating out the A^+ component which appears quadratically, the final lightcone theory contains only the two physical components A_z and $A_{\bar z}$ [64–66], which we associate with positive and negative helicity respectively. The Lagrangian takes the simple form (2.1)

$$\mathcal{L}_{\rm YM} = \mathcal{L}_{+-} + \mathcal{L}_{++-} + \mathcal{L}_{--+} + \mathcal{L}_{++--}\,, \tag{A.16}$$

with

$$
\begin{aligned}
\mathcal{L}_{+-} &= -2\,\mathrm{tr}\{A_{\bar z}(\partial_+\partial_- - \partial_z\partial_{\bar z})A_z\}\,,\\
\mathcal{L}_{++-} &= 2ig\,\mathrm{tr}\{[A_z,\partial_+A_{\bar z}](\partial_+)^{-1}(\partial_{\bar z}A_z)\}\,,\\
\mathcal{L}_{--+} &= 2ig\,\mathrm{tr}\{[A_{\bar z},\partial_+A_z](\partial_+)^{-1}(\partial_z A_{\bar z})\}\,,\\
\mathcal{L}_{++--} &= -2g^2\,\mathrm{tr}\{[A_{\bar z},\partial_+A_z](\partial_+)^{-2}[A_z,\partial_+A_{\bar z}]\}\ .
\end{aligned}
\tag{A.17}
$$

Note that, in agreement with CQT, we have used the normalisation $\mathrm{tr}\{T^aT^b\} = \delta^{ab}$. In order to convert to the usual conventions for Yang-Mills theory, we therefore need to rescale $g \to g/\sqrt{2}$.

Relation to the notation of CQT. To compare our notation to that of [39 – 41], note that we employ outgoing momenta instead of incoming, therefore the all-plus amplitudes in these works would be all-minus from our perspective, and should thus be conjugated when comparing. Also, our time evolution coordinate is taken to be x^- rather that x^+, which (among other changes) implies that p^+ of CQT becomes p_+. Our metric is also taken to have opposite signature to that in CQT. Finally, CQT define momentum brackets $K^{\wedge}_{ij}$ and $K^{\vee}_{ij}$, which are just our (ij) and $\{ij\}$ brackets respectively.

B. Details on the four-point calculation

In this appendix we prove two results that were used in section 3, namely equations (3.11) and (3.15). To make the expressions more compact, instead of momentum brackets we use the following notation:

$$
f_{ij} = -\frac{(ij)}{p_+^i p_+^j} = \frac{p_z^i}{p_+^i} - \frac{p_z^j}{p_+^j}\ . \tag{B.1}
$$

The f_{ij} variables satisfy the simple relation:

$$
f_{ij} = f_{ik} + f_{kj}\,, \tag{B.2}
$$

while momentum conservation is applied as

$$
p_+^i f_{ij} = -\sum p_+^k f_{kj}\ . \tag{B.3}
$$

Also, to minimise clutter, in this appendix we use the notation $q_i := p_+^i$.

Proof of the quadratic identity. In order to show (3.11), it is convenient to divide out by the $\sqrt{p_+^1p_+^2p_+^3p_+^4}$ factor (which is there anyway in (3.10)) in order to bring it to the form

$$
\begin{aligned}
&q_4^2f_{34}f_{41} + q_1^2f_{12}f_{41} + q_2^2f_{12}f_{23} + q_3^2f_{23}f_{34}\\
&- (q_2+q_3)(q_1+q_4)f_{12}f_{34} - (q_3+q_4)(q_2+q_1)f_{23}f_{41} = 0\,,
\end{aligned}
\tag{B.4}
$$

Expanding out the two last terms in (B.4) as

$$
-(q_1q_3+q_2q_4)(f_{12}f_{34}+f_{23}f_{41}) - (q_1q_2+q_3q_4)f_{12}f_{34} - (q_2q_3+q_4q_1)f_{23}f_{41}\ , \tag{B.5}
$$

we apply momentum conservation on each of the four components of the first term of (B.5), in the following way:

$$
\begin{aligned}
-\, q_1 q_3 f_{12} f_{34} &= q_1 f_{12}(q_1 f_{14} + q_2 f_{24}) = -q_1^2 f_{12} f_{41} + q_1 q_2 f_{12} f_{24} \,, \\
-\, q_1 q_3 f_{23} f_{41} &= q_3 f_{23}(q_2 f_{42} + q_3 f_{43}) = -q_3^2 f_{23} f_{34} + q_2 q_3 f_{23} f_{42} \,, \\
-\, q_2 q_4 f_{12} f_{34} &= q_4(q_3 f_{13} + q_4 f_{14}) f_{34} = -q_4^2 f_{34} f_{41} + q_3 q_4 f_{13} f_{34} \,, \\
-\, q_2 q_4 f_{23} f_{41} &= q_2 f_{23}(q_2 f_{21} + q_3 f_{31}) = -q_2^2 f_{12} f_{23} + q_2 q_3 f_{31} f_{23} \,.
\end{aligned}
\tag{B.6}
$$

Clearly these transformations have been chosen to cancel the first four terms in (B.4). Collecting the remaining terms, we obtain

$$
\begin{aligned}
& q_1 q_2 f_{12}(f_{24} - f_{34}) + q_2 q_3 f_{23}(f_{42} + f_{31} - f_{41}) + q_3 q_4 f_{34}(f_{13} - f_{12}) - q_1 q_4 f_{23} f_{41} \\
&= q_1 q_2 f_{12} f_{23} + q_2 q_3 f_{23} f_{32} + q_3 q_4 f_{34} f_{23} + q_1 q_4 f_{23} f_{14} \\
&= f_{23}[q_2(q_1 f_{12} + q_3 f_{32}) + q_4(q_3 f_{34} + q_1 f_{14})] = f_{23}[-q_2(q_4 f_{42}) - q_4(q_2 f_{24})] \\
&= 0
\end{aligned}
\tag{B.7}
$$

thus showing (3.11).

Proof of the linear identity. We will now outline the proof ot the linear (in region momenta) identity (3.15). Converting it to the notation used in the appendix, and performing simple manipulations, we find (suppressing the overall 3/8 factor):

$$
\begin{aligned}
X = {} & q_4^2((\bar{p}_3 + \bar{p}_4) + (\bar{p}_2 + \bar{p}_3)) f_{34} f_{41} + q_1^2(-(\bar{p}_2 + \bar{p}_3) + (\bar{p}_3 + \bar{p}_4)) f_{12} f_{41} \\
& + q_2^2(-(\bar{p}_3 + \bar{p}_4) - (\bar{p}_2 + \bar{p}_3)) f_{12} f_{23} + q_3^2(+(\bar{p}_2 + \bar{p}_3) - (\bar{p}_3 + \bar{p}_4)) f_{23} f_{34} \\
& - \frac{1}{2}(q_2 + q_3)(q_1 + q_4)[(\bar{p}_3 - \bar{p}_2) + (\bar{p}_1 - \bar{p}_4)] f_{12} f_{34} \\
& - \frac{1}{2}(q_3 + q_4)(q_1 + q_2)[(\bar{p}_4 - \bar{p}_3) + (\bar{p}_2 - \bar{p}_1)] f_{23} f_{41} \\
= {} & (\bar{p}_3 - \bar{p}_1)(q_4^2 f_{34} f_{41} - q_2^2 f_{12} f_{23}) + (\bar{p}_4 - \bar{p}_2)(q_1^2 f_{12} f_{41} - q_3^2 f_{23} f_{34}) \\
& - (q_2 + q_3)(q_1 + q_4)(\bar{p}_3 + \bar{p}_1) f_{12} f_{34} - (q_3 + q_4)(q_1 + q_2)(\bar{p}_2 + \bar{p}_4) f_{23} f_{41} \\
= {} & (\bar{p}_3 - \bar{p}_1)(q_4^2 f_{34} f_{41} - q_2^2 f_{12} f_{23}) + (\bar{p}_4 - \bar{p}_2)(q_1^2 f_{12} f_{41} - q_3^2 f_{23} f_{34}) \\
& - (\bar{p}_1 + \bar{p}_3) q_2 q_4 (f_{12} f_{34} - f_{23} f_{41}) + (\bar{p}_2 + \bar{p}_4) q_1 q_3 (f_{12} f_{34} - f_{23} f_{41}) \\
& - (\bar{p}_1 + \bar{p}_3)(q_1 q_2 + q_3 q_4) f_{12} f_{34} + (\bar{p}_1 + \bar{p}_3)(q_2 q_3 + q_4 q_1) f_{23} f_{41} \,.
\end{aligned}
\tag{B.8}
$$

Similarly to the previous case, we will rewrite the second line in the final expression in such a way that we completely cancel all the terms in the first line. To do that we use

$$
\begin{aligned}
-(\bar{p}_1 + \bar{p}_3) q_2 q_4 (f_{12} f_{34} - f_{23} f_{41}) = {} & (\bar{p}_3 - \bar{p}_1)(q_2^2 f_{12} f_{23} - q_4^2 f_{34} f_{41}) + \\
& + q_1 q_2 \bar{p}^1 f_{12} f_{31} - q_4 q_1 \bar{p}^1 f_{41} f_{13} + \\
& + q_3 q_4 \bar{p}^3 f_{34} f_{13} - q_2 q_3 \bar{p}^3 f_{23} f_{31}
\end{aligned}
\tag{B.9}
$$

and

$$
\begin{aligned}
(\bar{p}_2 + \bar{p}_4) q_1 q_3 (f_{12} f_{34} - f_{23} f_{41}) = {} & (\bar{p}_4 - \bar{p}_2)(q_3^2 f_{23} f_{34} - q_1^2 f_{12} f_{41}) + \\
& + q_2 q_3 \bar{p}_2 f_{23} f_{42} - q_1 q_2 \bar{p}_2 f_{12} f_{24} + \\
& + q_4 q_1 \bar{p}_4 f_{41} f_{24} - q_3 q_4 \bar{p}_4 f_{34} f_{42} \,.
\end{aligned}
\tag{B.10}
$$

What remains after substituting these is

$$
\begin{aligned}
X = &\ \bar{p}_1 q_1 f_{31}(q_2 f_{12} + q_4 f_{41}) + q_3 \bar{p}_3 f_{13}(q_4 f_{34} + q_2 f_{23}) \\
&+ \bar{p}_2 q_2 f_{42}(q_3 f_{23} + q_1 f_{12}) + q_4 \bar{p}_4 f_{24}(q_1 f_{41} + q_3 f_{34}) \\
&- (\bar{p}_1 + \bar{p}_3)(q_1 q_2 + q_3 q_4) f_{12} f_{34} + (\bar{p}_1 + \bar{p}_3)(q_2 q_3 + q_4 q_1) f_{23} f_{41} \\
= &\ \bar{p}_1 q_1 q_2 f_{12} f_{41} + \bar{p}_3 q_3 q_4 f_{34} f_{23} + \bar{p}_1 q_4 q_1 f_{41} f_{21} + \bar{p}_3 q_2 q_3 f_{23} f_{43} \\
&+ \bar{p}_2 q_2 f_{42}(q_3 f_{23} + q_1 f_{12}) + q_4 \bar{p}_4 f_{24}(q_1 f_{41} + q_3 f_{34}) \\
&- (\bar{p}_1 q_3 q_4 + \bar{p}_3 q_1 q_2) f_{12} f_{34} + (\bar{p}_1 q_2 q_3 + \bar{p}_3 q_4 q_1) f_{23} f_{41} \ .
\end{aligned} \tag{B.11}
$$

Now we collect various terms together to rewrite X as

$$
\begin{aligned}
X = &\ \bar{p}_1 q_2 f_{41}(q_1 f_{12} + q_3 f_{23}) + \bar{p}_3 q_4 f_{23}(q_3 f_{34} + q_1 f_{41}) \\
&+ \bar{p}_1 q_4 f_{21}(q_1 f_{41} + q_3 f_{34}) + \bar{p}_3 q_2 f_{43}(q_3 f_{23} + q_1 f_{12}) \\
&+ \bar{p}_2 q_2 f_{42}(q_3 f_{23} + q_1 f_{12}) + \bar{p}_4 q_4 f_{24}(q_1 f_{41} + q_3 f_{34}) \\
= &\ \bar{p}_1 q_2 f_{41}(2 q_3 f_{23} - q_4 f_{42}) + \bar{p}_3 q_4 f_{23}(2 q_1 f_{41} - q_2 f_{24}) \\
&+ \bar{p}_1 q_4 f_{21}(2 q_1 f_{41} - q_4 f_{42}) + \bar{p}_3 q_2 f_{43}(2 q_3 f_{23} - q_4 f_{42}) \\
&+ \bar{p}_2 q_2 f_{42}(2 q_3 f_{23} - q_4 f_{42}) + \bar{p}_4 q_4 f_{24}(2 q_1 f_{41} - q_2 f_{24}) \\
= &\ 2[q_2 q_3 f_{23}(\bar{p}_1 f_{41} + \bar{p}_3 f_{43} + \bar{p}_2 f_{42}) + q_4 q_1 f_{41}(\bar{p}_3 f_{23} + \bar{p}_1 f_{21} + \bar{p}_4 f_{24})] \\
&+ (\bar{p}_1 + \bar{p}_2 + \bar{p}_3 + \bar{p}_4) q_2 q_4 f_{24} f_{42} \ .
\end{aligned} \tag{B.12}
$$

Clearly the term on the last line vanishes by momentum conservation. We now restore all labels to write the final result as

$$
\begin{aligned}
X = &2\ (32)[f_4(p^1_{\bar{z}} + p^2_{\bar{z}} + p^3_{\bar{z}}) - p^1_{\bar{z}} f_1 - p^2_{\bar{z}} f_2 - p^3_{\bar{z}} f_3] + \\
&+ 2\ (14)[f_2(p^1_{\bar{z}} + p^3_{\bar{z}} + p^4_{\bar{z}}) - p^3_{\bar{z}} f_3 - p^1_{\bar{z}} f_1 - p^4_{\bar{z}} f_4] \,,
\end{aligned} \tag{B.13}
$$

where we used that $q_2 q_3 f_{23} = p^2_+ p^3_+ (p^2_z/q^2_+ - p^3_z/p^3_+) = p^3_+ p^2_z - p^2_+ p^3_z = (32)$ (and similarly for (14)), and where $f_i = p^i_z/p^i_+$. Using momentum conservation on both terms, we rewrite them as

$$
X = -2[(32) + (14)] \left[\frac{p^1_{\bar{z}} p^1_z}{p^1_+} + \frac{p^2_{\bar{z}} p^2_z}{p^2_+} + \frac{p^3_{\bar{z}} p^3_z}{p^3_+} + \frac{p^4_{\bar{z}} p^4_z}{p^4_+} \right] \ . \tag{B.14}
$$

For each momentum we have that $p^2 = 2(p_+ p_- - p_z p_{\bar{z}})$, therefore we can rewrite the above as

$$
X = +[(32) + (14)] \left[\frac{(p_1)^2}{p^1_+} + \frac{(p_2)^2}{p^2_+} + \frac{(p_3)^2}{p^3_+} + \frac{(p_4)^2}{p^4_+} + 2(p^1_- + p^2_- + p^3_- + p^4_-) \right] \ . \tag{B.15}
$$

The p_- term vanishes, hence, noticing also that $(32) + (14) = -\frac{1}{2}((12) + (23) + (34) + (41))$, we conclude that

$$
X = -\frac{1}{2}[(12) + (23) + (34) + (41)] \sum_{i=1}^{4} \frac{(p_i)^2}{p^i_+} \ . \tag{B.16}
$$

Off-shell terms in the four-point case. For completeness, we also give the form of the off-shell terms that arose in the manipulations leading to (3.26).

Using the notation $P_{ij} = (\frac{(p_i)^2}{p^i_+} + \frac{(p_j)^2}{p^j_+})$ they are:

$$
\begin{aligned}
f(p^2) = & \frac{1}{4\langle 12\rangle \cdots \langle 41\rangle}\Bigg[- P_{13}(\bar{p}_1+\bar{p}_2)(41) - P_{13}(\bar{p}_2+\bar{p}_3)(12) + P_{24}(\bar{p}_2+\bar{p}_3)(42) \\
& + \frac{1}{p^1_+}P_{12}[(p^2_+ + p^3_+)(2\bar{p}_1+\bar{p}_2-\bar{p}_3) - p^3_+(\bar{p}_1+\bar{p}_2) - p^1_+(\bar{p}_2+\bar{p}_3)]\ (13) \\
& + P_{12}\frac{p^3_+}{p^1_+p^2_+}[p^2_+(\bar{p}_1+\bar{p}_2) - p^4_+(\bar{p}_2+\bar{p}_3)](12) - 2P_{13}\frac{1}{p^1_+}\{31\}(41)\Bigg] \,.
\end{aligned} \tag{B.17}
$$

This expression, together with $\mathcal{V}^{(4)}_k$ in (3.16), should be added to (3.26) in order to recover a fully off-shell four-point vertex.

References

[1] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [hep-th/0312171].

[2] F. Cachazo and P. Svrcek, *Lectures on twistor strings and perturbative Yang-Mills theory*, PoS(RTN2005)004 [hep-th/0504194].

[3] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys.* **B 715** (2005) 499 [hep-th/0412308].

[4] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [hep-th/0501052].

[5] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, *Nucl. Phys.* **B 725** (2005) 275 [hep-th/0412103].

[6] A. Brandhuber and G. Travaglini, *Quantum MHV diagrams*, hep-th/0609011.

[7] F. Cachazo, P. Svrcek and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *JHEP* **09** (2004) 006 [hep-th/0403047].

[8] G. Georgiou and V.V. Khoze, *Tree amplitudes in gauge theory as scalar MHV diagrams*, *JHEP* **05** (2004) 070 [hep-th/0404072].

[9] C.-J. Zhu, *The googly amplitudes in gauge theory*, *JHEP* **04** (2004) 032 [hep-th/0403115].

[10] J.-B. Wu and C.-J. Zhu, *Mhv vertices and scattering amplitudes in gauge theory*, *JHEP* **07** (2004) 032 [hep-th/0406085].

[11] J.-B. Wu and C.-J. Zhu, *Mhv vertices and fermionic scattering amplitudes in gauge theory with quarks and gluinos*, *JHEP* **09** (2004) 063 [hep-th/0406146].

[12] G. Georgiou, E.W.N. Glover and V.V. Khoze, *Non-MHV tree amplitudes in gauge theory*, *JHEP* **07** (2004) 048 [hep-th/0407027].

[13] L.J. Dixon, E.W.N. Glover and V.V. Khoze, *MHV rules for Higgs plus multi-gluon amplitudes*, *JHEP* **12** (2004) 015 [hep-th/0411092].

[14] Z. Bern, D. Forde, D.A. Kosower, and P. Mastrolia, *Twistor-inspired construction of electroweak vector boson currents*, *Phys. Rev.* **D 72** (2005) 025006 [hep-ph/0412167].

[15] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *Non-supersymmetric loop amplitudes and MHV vertices*, *Nucl. Phys.* **B 712** (2005) 59 [hep-th/0412108].

[16] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys.* **B 435** (1995) 59 [hep-ph/9409265].

[17] A. Brandhuber, B.J. Spence and G. Travaglini, *One-loop gauge theory amplitudes in $N = 4$ super Yang-Mills from mhv vertices*, *Nucl. Phys.* **B 706** (2005) 150 [hep-th/0407214].

[18] W.L. van Neerven, *Dimensional regularization of mass and infrared singularities in two loop on-shell vertex functions*, *Nucl. Phys.* **B 268** (1986) 453.

[19] Z. Bern and A.G. Morgan, *Massive loop amplitudes from unitarity*, *Nucl. Phys.* **B 467** (1996) 479 [hep-ph/9511336].

[20] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One-loop self-dual and $N = 4$ super Yang-Mills*, *Phys. Lett.* **B 394** (1997) 105 [hep-th/9611127].

[21] A. Brandhuber, S. McNamara, B.J. Spence and G. Travaglini, *Loop amplitudes in pure Yang-Mills from generalised unitarity*, *JHEP* **10** (2005) 011 [hep-th/0506068].

[22] Z. Bern, L.J. Dixon and D.A. Kosower, *Bootstrapping multi-parton loop amplitudes in QCD*, *Phys. Rev.* **D 73** (2006) 065013 [hep-ph/0507005].

[23] C.F. Berger, Z. Bern, L.J. Dixon, D. Forde and D.A. Kosower, *Bootstrapping one-loop QCD amplitudes with general helicities*, *Phys. Rev.* **D 74** (2006) 036009 [hep-ph/0604195].

[24] C.F. Berger, Z. Bern, L.J. Dixon, D. Forde and D.A. Kosower, *All one-loop maximally helicity violating gluonic amplitudes in QCD*, *Phys. Rev.* **D 75** (2007) 016006 [hep-ph/0607014].

[25] Z. Xiao, G. Yang and C.-J. Zhu, *The rational part of QCD amplitude. I: the general formalism*, *Nucl. Phys.* **B 758** (2006) 1 [hep-ph/0607015].

[26] X. Su, Z. Xiao, G. Yang and C.-J. Zhu, *The rational part of QCD amplitude. ii: the five-gluon*, *Nucl. Phys.* **B 758** (2006) 35 [hep-ph/0607016].

[27] Z. Xiao, G. Yang and C.-J. Zhu, *The rational part of QCD amplitude. III: the six-gluon*, *Nucl. Phys.* **B 758** (2006) 53 [hep-ph/0607017].

[28] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *D-dimensional unitarity cut method*, *Phys. Lett.* **B 645** (2007) 213 [hep-ph/0609191].

[29] P. Mastrolia, *On triple-cut of scattering amplitudes*, *Phys. Lett.* **B 644** (2007) 272 [hep-th/0611091].

[30] R. Britto and B. Feng, *Unitarity cuts with massive propagators and algebraic expressions for coefficients*, *Phys. Rev.* **D 75** (2007) 105006 [hep-ph/0612089].

[31] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *Unitarity cuts and reduction to master integrals in d dimensions for one-loop amplitudes*, *JHEP* **03** (2007) 111 [hep-ph/0612277].

[32] A. Gorsky and A. Rosly, *From Yang-Mills lagrangian to MHV diagrams*, *JHEP* **01** (2006) 101 [hep-th/0510111].

[33] P. Mansfield, *The lagrangian origin of MHV rules*, *JHEP* **03** (2006) 037 [hep-th/0511264].

[34] J.H. Ettle and T.R. Morris, *Structure of the MHV-rules lagrangian*, *JHEP* **08** (2006) 003 [hep-th/0605121].

[35] A. Brandhuber, B. Spence and G. Travaglini, *Amplitudes in pure Yang-Mills and MHV diagrams*, *JHEP* **02** (2007) 088 [hep-th/0612007].

[36] D. Cangemi, *Self-dual Yang-Mills theory and one-loop maximally helicity violating multi-gluon amplitudes*, *Nucl. Phys.* **B 484** (1997) 521 [hep-th/9605208].

[37] D. Cangemi, *Self-duality and maximally helicity violating QCD amplitudes*, *Int. J. Mod. Phys.* **A 12** (1997) 1215 [hep-th/9610021].

[38] G. Chalmers and W. Siegel, *The self-dual sector of QCD amplitudes*, *Phys. Rev.* **D 54** (1996) 7628 [hep-th/9606061].

[39] C.B. Thorn, *Notes on one-loop calculations in light-cone gauge*, hep-th/0507213.

[40] D. Chakrabarti, J. Qiu and C.B. Thorn, *Scattering of glue by glue on the light-cone worldsheet. I: helicity non-conserving amplitudes*, *Phys. Rev.* **D 72** (2005) 065022 [hep-th/0507280].

[41] D. Chakrabarti, J. Qiu and C.B. Thorn, *Scattering of glue by glue on the light-cone worldsheet. II: helicity conserving amplitudes*, *Phys. Rev.* **D 74** (2006) 045018 [hep-th/0602026].

[42] H. Feng and Y.-t. Huang, *MHV lagrangian for $N = 4$ super Yang-Mills*, hep-th/0611164.

[43] A. Brandhuber, B. Spence and G. Travaglini, *From trees to loops and back*, *JHEP* **01** (2006) 142 [hep-th/0510253].

[44] K. Bardakci and C.B. Thorn, *A worldsheet description of large-N_c quantum field theory*, *Nucl. Phys.* **B 626** (2002) 287 [hep-th/0110301].

[45] C.B. Thorn, *A worldsheet description of planar Yang-Mills theory*, *Nucl. Phys.* **B 637** (2002) 272 [hep-th/0203167].

[46] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[47] H.B. Nielsen and P. Olesen, *A parton view on dual amplitudes*, *Phys. Lett.* **B 32** (1970) 203.

[48] B. Sakita and M.A. Virasoro, *Dynamical model of dual amplitudes*, *Phys. Rev. Lett.* **24** (1970) 1146.

[49] C.B. Thorn, *Renormalization of quantum fields on the lightcone worldsheet. I: scalar fields*, *Nucl. Phys.* **B 699** (2004) 427 [hep-th/0405018].

[50] K. Bardakci and C.B. Thorn, *A mean field approximation to the world sheet model of planar ϕ^3 field theory*, *Nucl. Phys.* **B 652** (2003) 196 [hep-th/0206205].

[51] K. Bardakci, *Self consistent field method for planar ϕ^3 theory*, *Nucl. Phys.* **B 677** (2004) 354 [hep-th/0308197].

[52] Z. Bern, L. Dixon, D. Dunbar, and D. Kosower, *One-loop n-point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-th/9403226].

[53] J. Qiu, *One loop gluon gluon scattering in light cone gauge*, *Phys. Rev.* **D 74** (2006) 085022 [hep-th/0607097].

[54] Z. Bern and D.A. Kosower, *The computation of loop amplitudes in gauge theories*, *Nucl. Phys.* **B 379** (1992) 451.

[55] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, *JHEP* **02** (2000) 020 [hep-th/9912072].

[56] Z. Bern, G. Chalmers, L.J. Dixon and D.A. Kosower, *One loop n gluon amplitudes with maximal helicity violation via collinear limits*, *Phys. Rev. Lett.* **72** (1994) 2134 [hep-ph/9312333].

[57] F. Cachazo, P. Svrček and E. Witten, *Twistor space structure of one-loop amplitudes in gauge theory*, *JHEP* **10** (2004) 074 [hep-th/0406177].

[58] R. Boels, L. Mason and D. Skinner, *Supersymmetric gauge theories in twistor space*, *JHEP* **02** (2007) 014 [hep-th/0604040].

[59] R. Boels, L. Mason and D. Skinner, *From twistor actions to MHV diagrams*, *Phys. Lett.* **B 648** (2007) 90 [hep-th/0702035].

[60] R. Boels, *A quantization of twistor Yang-Mills theory through the background field method*, hep-th/0703080.

[61] J.H. Ettle, C.-H. Fu, J.P. Fudger, P.R.W. Mansfield and T.R. Morris, *S-matrix equivalence theorem evasion and dimensional regularisation with the canonical MHV lagrangian*, hep-th/0703286.

[62] D.A. Kosower, *Next-to-maximal helicity violating amplitudes in gauge theory*, *Phys. Rev.* **D 71** (2005) 045007 [hep-th/0406175].

[63] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *A twistor approach to one-loop amplitudes in $N = 1$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 706** (2005) 100 [hep-th/0410280].

[64] E. Tomboulis, *Quantization of the Yang-Mills field in the null-plane frame*, *Phys. Rev.* **D 8** (1973) 2736.

[65] J. Scherk and J.H. Schwarz, *Gravitation in the light cone gauge*, *Gen. Rel. Grav.* **6** (1975) 537.

[66] D.M. Capper, J.J. Dulwich and M.J. Litvak, *On the evaluation of integrals in the light cone gauge*, *Nucl. Phys.* **B 241** (1984) 463.